<u>Definition</u> : A vector field $\vec{F}$ is called "rotation-free" if $\text{curl}(\vec{F}) = \underline{0}$.

and "incompressible" if $\text{div}(\vec{F}) = \underline{0}$.

check the following properties (consequences of fact that $d(d\varphi) = 0$ or just check directly ...)

① $\text{curl}(\text{grad}(f)) = 0$

② $\text{div}(\text{curl}(F)) = 0$

<u>Example</u> : a magnetic field is always expressible as $\text{curl}(\vec{A})$ for some vector field $\vec{A}$. Thus, magnetic field is always incompressible.

Examples of Stokes' theorem : $\underset{\text{compact, "good" boundary}}{\underbrace{X}} \subseteq M$: k-manifold $\subset \mathbb{R}^n$, φ : k-1-form

then $$\int_{\partial X} \varphi = \int_X d\varphi.$$

Easy example : X = rectangle in $\mathbb{R}^2$. Then $\int_{\partial X} \varphi$, φ : 1-form

say with vertices $(0,0), (a,0), (0,b), (a,b)$

is painful as # of sides in boundary is $2n = 4$. (worse in higher dimensions)

Pick φ : $x\, dy - dx$

$$d\varphi = d(x\,dy - dx) = dx \wedge dy - d(dx) = \underbrace{dx \wedge dy}_{\text{volume form on } \mathbb{R}^2}.$$

so by Stokes' thm.

$$\int_{\partial X} \varphi = \int_X d\varphi = \int_{\mathcal{U}} dx \wedge dy = a \cdot b.$$

expand this:

γ : param. of rectangle.

$$\int_{\mathcal{U}} \operatorname{Det}(D\gamma(\underline{u}))|d^k \underline{u}| = \int_M |d^k \underline{x}|.$$

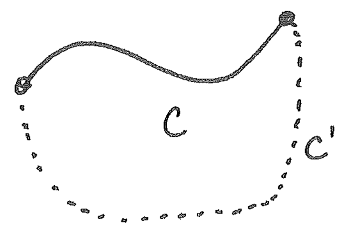
Harder example over cube in $\mathbb{R}^3$, integrating 2-form over boundary of cube.

or can pick "trivial param"

$$\gamma : \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$(u_1, u_2) \in$ Rectangle.

Even if want to integrate over (k-1)-manifold not a boundary, still play tricks: e.g. want to integrate over C, complete it to boundary of 2-manifold using C', where C' simple enough.



$C \cup C' = \partial X$

X: "good" 2-manifold

Stokes' theorem gives $$\int_{\partial X} \varphi = \int_X d\varphi$$

So $$\int_C \varphi = \int_X d\varphi - \int_{C'} \varphi.$$

For example, if $d\varphi = 0$ then $$\int_C \varphi = -\underbrace{\int_{C'} \varphi}$$

C' any curve with same endpoints. (C' nice enough)

e.g. $\varphi = x\,dy + y\,dx$

then $d\varphi = dx \wedge dy + dy \wedge dx = 0.$

(pick 0-form f, then df is 1-form and $d(df) = 0$)

$f = xy$ then $df = dx \cdot y + dy\,x.$

Earlier sketch of Stokes' theorem:

$$\int_X d\varphi \approx \sum_{i=1}^{N} \underbrace{d\varphi(P_i)}_{\substack{\text{assume } \varphi \\ \text{constant on} \\ \text{small } P_i}} \approx \sum_{i=1}^{N} \int_{\partial P_i} \varphi \approx \int_{\partial X} \varphi$$

definition of d as flux.

orientations on boundaries cancel.

Works nicely if boundary of X is well-approximated by dyadic paving.