On Friday, defined exterior derivative as flux: φ : $k-1$-form

$$d\varphi(\underline{x})(v_1, \dots, v_k) = \lim_{h \to 0} \frac{1}{h^k} \int_{\partial P_{\underline{x}}(hv_1, \dots, hv_k)} \varphi$$

so that Stokes' theorem (at least approximately) appeared true.

Owes: show $d\varphi$ computable.

Trying to show $d(f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

Plan: use definition, pairing opposite sides of $\partial P_{\underline{x}}$ (which have opposite orientation)

and Taylor expansion of f (with $\underline{x}$ moved to $\underline{0}$)

two parametrizations

Key pt.: Derivative same on both sides in vars $\neq i$.

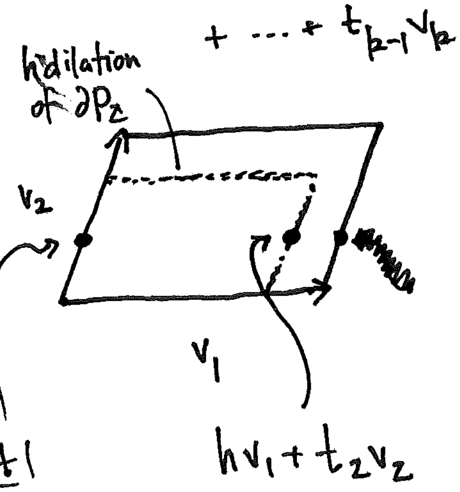
linear terms of f (with orientation on each boundary)

$$\gamma_{0,i}(\underline{t}) = t_1 v_1 + \cdots + 0 v_i + \cdots t_{k-1} v_k$$

$$[Df(0)](\underbrace{h\underline{v}_i + \gamma_{0,i}(\underline{t})}_{\gamma_{1,i}(\underline{t})}) - [Df(0)](\gamma_{0,i}(\underline{t}))$$

$$\gamma_{1,i}(\underline{t}) = t_1 v_1 + \cdots + h v_i + \cdots + t_{k-1} v_k$$

$$= h \cdot [Df(\underline{0})]\, \underline{v}_i.$$



so plugging into definition of exterior deriv:

$$\lim_{h \to 0} \frac{1}{h^k} \int_{[t_1, \dots, t_{k-1}] \in [0,1]^{k-1}} \left| f(\gamma_{1,i}(\underline{t})) - f(\gamma_{0,i}(\underline{t})) \right| \, dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}}(D\gamma(\underline{t}))\, |d^k \underline{t}|$$

this is const.! $\begin{pmatrix} v_j\text{'s} \\ j \neq i \end{pmatrix}$

Return to connection to physics. Favorite forms in $\mathbb{R}^3$.

Work form $\quad W_F(\underline{x})(\underline{v}) = F(\underline{x}) \cdot \underline{v} \quad$ (1-form)

Flux form $\quad \Phi_F(\underline{x})(\underline{v},\underline{w}) = \det \begin{bmatrix} | & | & | \\ F(\underline{x}) & \underline{v} & \underline{w} \\ | & | & | \end{bmatrix}$

(+ 0-forms (functions) and 3-forms (volume via det.))

integral: evaluation $\qquad$ integral: "mass" – weighted sum of volumes.

Natural question: What is effect of exterior derivative on favorite forms?

Given 0-form f, apply df. It's a 1-form and all 1-forms are W_g for some g

So what is the g? $\qquad (g_1 dx_1 + \cdots + g_3 dx_3)$

Ans: $\quad df = W_{\vec{\nabla} f}$

where $\vec{\nabla} f$ = "gradient of f" or write $\operatorname{grad}(f) := \begin{bmatrix} D_1 f \\ D_2 f \\ D_3 f \end{bmatrix}$

(equally true in $\mathbb{R}^n$, going from 0-forms to 1-forms)

$g(\underline{x}) \det(\cdot) = g(\underline{x})\, dx_1 \wedge dx_2 \wedge dx_3$

Given 2-form Φ_F, then $d\Phi_F$ = mult. of det (call it M_g for "mass" form)

(just compute both sides symbolically)

$g = \vec{\nabla} \cdot \vec{F}$

$\vec{\nabla} \cdot \vec{F} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \cdot \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$

$= D_1 F_1 + D_2 F_2 + D_3 F_3$

"div(F)"

Same for (n−1)-form in $\mathbb{R}^n$ to n-form in $\mathbb{R}^n$.

In $\mathbb{R}^n$, physical intuition for 0, 1, n−1, n forms.

In $\mathbb{R}^3$, happens that $n-1 = 2$, so can try to express dW_F as flux form Φ_G

(do example with generic function...)

So what is G s.t. $dW_F = \Phi_G$? $G = \vec{\nabla} \times \vec{F}$ "curl of F"

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} D_2F_3 - D_3F_2 \\ D_3F_1 - D_1F_3 \\ D_1F_2 - D_2F_1 \end{bmatrix}$$

example is a little singular since only in $\mathbb{R}^3$ that we can get from work to flux.

What is physical intuition behind div, grad, curl?

Gradient form $W_{\nabla f}(\underline{x})(\underline{v}) = [Df(\underline{x})] \cdot \underline{v}$

This is directional derivative of f in direction of $\underline{v}$.

It is fastest (biggest) when v points in same direction as $[Df(\underline{x})]$

so $[Df(\underline{x})]$ = "grad $(f(\underline{x}))$" points in direction of fastest increase.

What about div? Just fall back on definition of d as flux.

$$(\operatorname{div} F) \cdot dx \wedge dy \wedge dz = \underbrace{d\Phi_F}_{\text{flux in small box around } \underline{x}}$$

Curl is more painful to explain...

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other useful properties we didn't mention

$d(d\varphi) = 0$ φ k-form, class C^2

$d(\varphi \wedge \psi) =$ ~~[illegible]~~ $d\varphi \wedge \psi + (-1)^k \varphi \wedge d\psi$

Definition: A vector field $\vec{F}$ is called "rotation-free" if $\text{curl}(\vec{F}) = \underline{0}$.

and "incompressible" if $\text{div}(\vec{F}) = \underline{0}$.

check the following properties (consequences of fact that $d(d\varphi) = 0$ or just check directly ...)

① $\text{curl}(\text{grad}(f)) = 0$

② $\text{div}(\text{curl}(F)) = 0$

Example: a magnetic field is always expressible as $\text{curl}(\vec{A})$ for some vector field $\vec{A}$. Thus, magnetic field is always incompressible.