Know how to integrate on oriented manifolds, want to end the semester with Stokes' theorem on manifolds – generalization of fundamental thm of calculus.

X compact subset of k-manifold M. $\quad \partial X$: boundary of X (roughly $k-1$ dimensional)

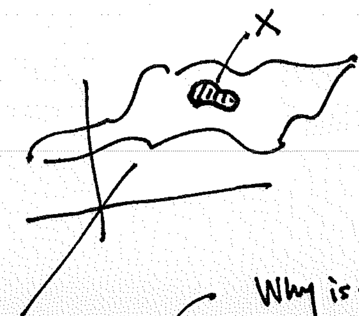
φ : $k-1$-form field.

then $$\int_{\partial X} \varphi = \int_X d\varphi$$

What is ∂X?

What is $d\varphi$?

Intuitive notion of ∂X:



Long, long ago defined boundary of a set U in $\mathbb{R}^n$ to be points u s.t. any nbhd. in $\mathbb{R}^n$ contains points of U, points not in U (i.e. in $\mathbb{R}^n \setminus U$)

Why is this definition bad in our situation?

M : k-manifold.

nbhds in $\mathbb{R}^n$ are n-dimensional.

Every point of our "wavy curtain" M would be boundary pt.

sphere cutting through "wavy curtain"

Need a definition relative to manifold M.

$X \subset M$, define ∂X to be points with each nbhd. of $\underline{x} \in M \subseteq \mathbb{R}^n$ (so n-dim'l ball) intersects both X and $M \setminus X$.

Another aspect of ∂X must be dealt with carefully: places where boundary not sufficiently differentiable.

How do we define it?

Any k-manifold M has, at each $\underline{x} \in M$, locally defined C^1-function

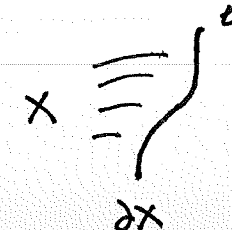
$F : \underset{\text{nbhd of } \underline{x}}{\underbrace{U}} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{n-k}$ with DF surjective at $\underline{x}$,

$M \cap U$ defined by $F = 0$.

Boundary, in order to be smooth $k-1$-manifold, should add one more condition: $\exists\ g : U \longrightarrow \mathbb{R}$ s.t. g is C^1 function AND

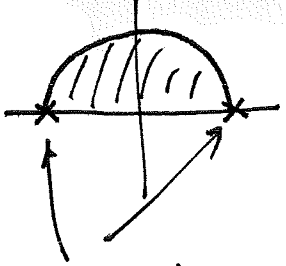
For $\underline{x} \in \partial_M X$ s.t. $g(\underline{x}) = 0$ and $X \cap U$ defined by $F = 0,\ g \geq 0$.

$+$ $D\begin{pmatrix} F \\ g \end{pmatrix}(\underline{x})$ surjective



want this set to be determined by $g \geq 0$, $g \in C^1(U)$.

Example: $\underset{X}{\underbrace{\text{Semi-circle}}}$ in $\underset{M}{\underbrace{\mathbb{R}^2}}$.
$X \subseteq M$



singular pts.

$$X = \{ (x,y) \in \mathbb{R}^2 \mid y \geq 0 \text{ and } x^2 + y^2 \leq 1 \}$$

check ∂X is as we expect,

and that there are two suspected singular points.

(For us, hard to check point is singular. Primary concern is that they have $(k-1)$-volume 0.)

Proposition: Let $\partial_M^S(X) :=$ set of smooth points on boundary $\partial_M(X)$.

then $\partial_M^S(X)$ is smooth $(k-1)$-dimensional manifold.

pf: Know $F = 0,\ g = 0$ defines smooth $(k-1)$-dimensional manifold since, by assumption, $D\begin{pmatrix} F \\ g \end{pmatrix}(\underline{x})$ surjective, so surjective on some open nbhd.

claim: $\{ \underline{y} \mid \underbrace{F(\underline{y}) = \underline{0},\ g(\underline{y}) = 0}_{Y} \} = \partial_M X \cap \underbrace{\text{nbhd. of } \underline{x} \text{ on which Jacobian of } F, g \text{ surjective.}}_{V}$

Since $V \cap X = \{ \underline{x} \in \underbrace{M}_{F=0} \cap V \mid g(\underline{x}) \geq 0 \}$

then $\partial_M X \cap V = \{ \underline{y} \in M \mid g(\underline{y}) = 0$ and for which g is pos./neg. in any nbhd. of $\underline{y} \}$

$= Y$ since the restriction of $[Dg(\underline{y})]$ to $T_{\underline{y}} M$ is surjective for every $\underline{y} \in Y$.

so far: know that 0-locus of $\binom{F}{g}$ gives manifold, but not that 0-locus matches boundary of X, $\partial_M X$.

in any nbhd of point.

so $Dg(\underline{y}) \neq 0$ rather defines line through 0 taking pos./neg. values.

Finally we can make definition of <u>good</u> subsets for Stokes' theorem.

Book calls them awkwardly "piece-with-boundary":

M: smooth k-manifold X: compact subset is <u>good</u> if

1. smooth boundary $\partial_M^s X$ has finite k-1 volume.
2. non-smooth boundary $\partial_M X \setminus \partial_M^s X$ has 0 $(k-1)$-volume.

<u>Important example</u>: k-parallelogram in $\mathbb{R}^n =: X$.

<u>Rough idea</u>: "faces" of k-parallelogram are k-1 dim'l hyperplanes.