Last time: Orientations on manifolds $\leftarrow$ continuously varying assignment of orientation to tangent space

(V for vector space, map $\Omega : \mathcal{B}_V \to \{\pm 1\}$)

Goal:

- Assign orientation to manifold
- Find a parametrization to do integration
- Determine whether parametrization preserves orientation.

Big theorem: If param. preserves orientation, then integral of k-form on oriented k-manifold is independent of choice of such parametrization.

(use same definition of integration as ever:

$$\int_M \varphi = \int_U \varphi(\gamma(u))(D\gamma(u))\,|d^k \underline{u}|. \quad)$$

How to assign orientation?

For curve, chose nowhere vanishing, continuously varying tangent vector $\underline{t}(\underline{x})$ for $\underline{x} \in M$.

then chose $\Omega_{\underline{x}}^{\underline{t}}(\underline{v}) \overset{\text{def}}{=} \operatorname{sgn}(\underline{t}(\underline{x}) \cdot \underline{v})$

Example: unit circle $x^2+y^2=1$.

then $DF = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$ and $T_{\underline{x}} M = \ker(DF) = \langle \underbrace{\begin{pmatrix} -y \\ x \end{pmatrix}} \rangle$

span of this vector at (x,y).

$\begin{pmatrix} -y \\ x \end{pmatrix}$ non-vanishing on unit circle (or any circle of radius R, $R>0$)

so defines orientation $\operatorname{sgn}((-y,x) \cdot \underline{v})$

for surface, find continuously varying vector $\underline{n}$ as function of $\underline{x} \in M$, with $\underline{n}(x)$ NOT in tangent space.

(book wants you to think of normal vector, though construction is more general)

Choose $\Omega^{\underline{n}(\underline{x})}_{x}(v_1, v_2) = \operatorname{sgn}\left(\det\left(\underbrace{\underline{n}(\underline{x})}_{\text{out of tangent space}}\ \underbrace{v_1 \quad v_2}_{\text{live in tangent space}}\right)\right)$

$(\underline{x}, v_1, v_2)$

together they define honest 3-parallelogram so has non-zero 3-volume.

Example: Pick $[DF]^T$ if

$DF(\underline{x})$ non-vanishing for all $\underline{x} \in M$

(3×1 vector since $F: \mathbb{R}^3 \to \mathbb{R}^1$.)

(in higher dimensions $DF^T: \mathbb{R}^n \to \mathbb{R}^{n-k}$, needs to be surjective at each $\underline{x}$ to give non-deg. n-parallelogram.)

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(3) Checking if parametrization preserves orientation: Say γ is orientation preserving if

$$\Omega(D\gamma(\underline{u})) = +1 \quad \forall\ u \in U - X$$

(γ: relaxed param $U \to M \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^k$, failing at set X of vol. 0)

Example: unit circle. $\gamma(u) = \begin{bmatrix} \sin u \\ \cos u \end{bmatrix}$.

Does it preserve orientation of $\Omega^{t}_{\underline{x}}(v) = \operatorname{sign}(\underline{t}(\underline{x}) \cdot v)$?

Compute $\underline{t}(\gamma(u)) \cdot \gamma'(u) = -1$. No!

Not a big issue in this example, which is orientation reversing.

More serious issue: spherical coords for unit sphere in $\mathbb{R}^3$

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} \longmapsto \begin{pmatrix} \cos\phi \cos\theta \\ \cos\phi \sin\theta \\ \sin\phi \end{pmatrix} \qquad \theta \in [0, \pi], \quad \phi \in [-\pi, \pi]$$

Choose Ω: $\det \begin{bmatrix} | & | & | \\ n(\underline{x}) & v_1 & v_2 \\ | & | & | \end{bmatrix}$ with $n(\underline{x}) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

then you can check $\det \left[n\left(\gamma\begin{pmatrix}\theta\\\phi\end{pmatrix}\right), D_1\gamma\begin{pmatrix}\theta\\\phi\end{pmatrix}, D_2\gamma\begin{pmatrix}\theta\\\phi\end{pmatrix} \right]$

$$= -\cos\phi$$

so doesn't respect orientation, since $\cos\phi$ oscillating.

How could this be? Parametrizations supposed to give orientations.

Problem: spherical coords are relaxed parametrization. (At north/south poles where $\phi = \pm \pi/2$, get all θ mapping to same point.)
(i.e. $\cos\phi = 0$)