Gauss' theorem

$$\text{Area}\,(D_r(\underline{p})) = \pi r^2 - \frac{\pi K(\underline{p})}{12} r^4 + \text{higher order terms in } r.$$

plan: write parametrization

$$\gamma : \begin{pmatrix} \rho \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ f\begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \end{pmatrix} \end{pmatrix}$$

(move $\underline{p}$ to $\underline{0}$ with simple initial change of coords)

$$U \longrightarrow D_r(\underline{0})$$

replace f with its Taylor poly. in best coordinates; choose span of tangent plane so that quad. terms are diagonal since symmetric matrices are always diagonalizable.

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}(ax^2 + by^2) + \text{higher order terms}$$

Revised result: Given f, we have

$$\gamma : \begin{pmatrix} \rho \\ \theta \end{pmatrix} \longmapsto \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ \frac{1}{2}(\rho^2 \cdot a \cdot \cos^2\theta + \rho^2 \cdot b \cdot \sin^2\theta) + o(\rho^2) \end{pmatrix}$$

show that

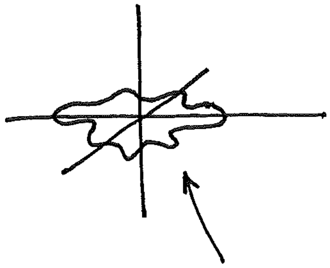
$$\text{Area}\,(D_r(\underline{0})) \overset{\text{def}}{=} \int_U \sqrt{\det\left[D\gamma\begin{pmatrix} \rho \\ \theta \end{pmatrix}^{T} D\gamma\begin{pmatrix} \rho \\ \theta \end{pmatrix}\right]}\, |d\rho\, d\theta| \;* = \pi r^2 - \frac{ab\pi}{12} r^4 + o(r^4)$$

Hard problem remains: Find open set U s.t.

$$\gamma(U) = D_r(\underline{0}).$$

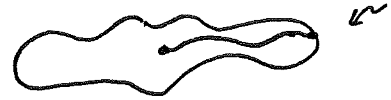
(or slightly less bad: $\gamma(U)$ closely approximates $D_r(\underline{0})$ since answer only approximate)

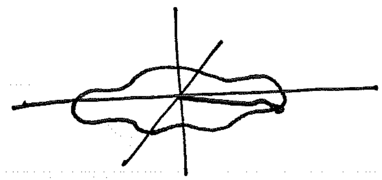
Rough picture:



projection of $D_r(\underline{0})$ onto x-y (aka (ρ, θ)) plane. We want parametric equation $\rho \le b(\theta)$ some function b.

In order to find / characterize the boundary of $D_r(\underline{0})$, only need to keep shortest paths from 0 to a point $\underline{x}$





paths from $\underline{0}$ to $\underline{x}$ are given as functions $\theta = h(\rho)$ some h.

Path : $s(\rho) = \begin{pmatrix} \rho \cos h(\rho) \\ \rho \sin h(\rho) \end{pmatrix}$

in x-y plane, with path lifted to surface

$$\tilde{s}(\rho) = \begin{pmatrix} \rho \cos h(\rho) \\ \rho \sin h(\rho) \\ f(s(\rho)) \end{pmatrix}$$

$s(0) = \underline{0}$

$s(r) = \begin{pmatrix} r \cos h(r) \\ r \sin h(r) \end{pmatrix}$

Guess: straight line paths in $\mathbb{R}^2$ lift to shorter paths in S.

Book calls straight line path

$$\delta_r(\rho) = \begin{pmatrix} \rho \cos h(r) \\ \rho \sin h(r) \end{pmatrix}$$

Proposition: for r suff. small, the lift of the straight line path

$\tilde{\delta}_r(\rho) = \begin{pmatrix} \delta_r(\rho) \\ f(\delta_r(\rho)) \end{pmatrix}$ is shorter

that lift of any other path to $\begin{pmatrix} r \cos h(r) \\ r \sin h(r) \end{pmatrix}$ $\tilde{s}(\rho)$

if, in Taylor expansion of $s(\rho)$,

$$s(\rho) = \theta_0 + k\rho + \frac{m}{2}\rho^2 + \cdots$$

with $k \ne 0$.

pf of proposition (sketch). Calculate approximate arc length for $\tilde{s}(\rho)$ using Taylor expansion for $h(\rho)$. ($\tilde{\delta}_r(\rho)$ is special case with $k=0$.)

$$\text{Arc length}\left(\tilde{s}(\rho),\ \rho\in[0,r]\right) = \int_0^r \sqrt{|s'(\rho)|^2 + |[Df(s(\rho))]\,s'(\rho)|^2}\ d\rho.$$

$$|s'(\rho)|^2 = (\cos h(\rho) - \rho \sin h(\rho)\cdot h'(\rho))^2 + (\sin h(\rho) + \rho\cos h(\rho)\, h'(\rho))^2$$

$$= 1 + \rho^2\cdot h'(\rho)^2 = 1 + k^2\rho^2 + o(\rho^2)$$ ← will integrate to $o(r^3)$.

Similar game for second term. Use Taylor poly for $h(\rho)$ and for f in Df.

leaves us with integrand $\sqrt{1+\underbrace{(\cdots)}_{\text{small if } \rho \text{ small}}}$ but $\sqrt{1+x} = 1+\frac{x}{2}+o(x)$ so substitute.

$$\text{Length}\left(\tilde{s}(\rho),\ \rho\in[0,r]\right) = r + \frac{r^3}{6}\left(k^2 + (a\cos^2\theta_0 + b\sin^2\theta_0)^2\right) + o(r^3) \qquad (*)$$

this expression gets smaller if $k=0$ (m didn't matter)

Now to finish problem, we see lifts of straight line paths $\delta_r(\rho)$ have

$$\underset{\rho\in[0,r]}{\text{arc length}}\left(\tilde{\delta}_r(\rho)\right) = r + \frac{r^3}{6}(\text{const.}) + o(r^3) \qquad \left(\text{from } k=0 \text{ in } (*)\right)$$

Given length r in any direction, get lifted path $r + r^3/6\cdot\text{const.}$

Ask reverse question, given path of length r upstairs up to $o(r^3)$ what is downstairs? Use inverse function. Don't know in general but $r + (\text{const.})\,r^3$ has inverse $r - \underset{\text{same}}{\text{const.}}\,r^3$.