We define k-manifold volume for relaxed parametrization

$$\gamma : U - X \longrightarrow M \subseteq \mathbb{R}^n \quad \text{by}$$

$$\mathrm{vol}_k(M) = \int_{U-X} \sqrt{\det\left(D\gamma(\underline{u})^T D\gamma(\underline{u})\right)} \; |d^k \underline{u}|$$

can also insert function here $f(\gamma(\underline{u}))$

and if $f \cdot \sqrt{\det(\cdots)}$ is integrable,

then can compute weighted volumes.

<u>theoretical consideration</u>:

Given two parametrizations, show they give same volume integral

(answer shouldn't depend on how we enumerate points on manifold.

case of curve: parametrization determines how fast you traverse curve.)

proof: given two parametrizations

$$\gamma_1 : U \longrightarrow M \qquad \gamma_2 : V \longrightarrow M \qquad U, V \subseteq \mathbb{R}^k$$

$$\gamma_1 = \gamma_2 \circ \underbrace{\gamma_2^{-1} \circ \gamma_1}$$

$\Phi : U \longrightarrow V$: "change of vars"

with $D\gamma_1(\underline{u}) = D\gamma_2(\Phi(\underline{u})) \cdot D\Phi(\underline{u})$ by chain rule. (*)

Given

$$\int_V \sqrt{\det\left(D\gamma_2(\underline{v})^T D\gamma_2(\underline{v})\right)} \; f(\gamma_2(\underline{v})) \; |d^k \underline{v}|$$

oops! $|\det(D\Phi(\underline{u}))|$

$$\overset{\text{change of vars.}}{=} \int_U \sqrt{\det\left(D\gamma_2 \circ \Phi(\underline{u})^T D\gamma_2 \circ \Phi(\underline{u})\right)} \; f(\underbrace{\gamma_2(\Phi(\underline{u}))}_{\gamma_1(\underline{u})}) \; |d^k \underline{u}|$$

Expand $|\det(D\Phi(u))| = \sqrt{\det D\Phi^T D\Phi}$, do some rearranging

apply chain rule. done.

May have noticed we wrote this as pair of perfect parametrizations, not relaxed ones.

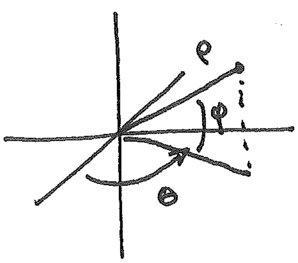
$\Phi : \mathcal{U} \to V$ must be bijective, class C^1, with Φ, Φ^{-1} having Lipschitz derivatives.

Book has nice example of issues at bad points in parametrization: two parametrizations of sphere using different poles ("north and south") (P_1, P_1') vs. (P_2, P_2')

$\Phi = \gamma_2^{-1} \circ \gamma_1$

if P_2, P_2' are poles of γ_2 then a single point maps to P_2 under γ_1.

so consider $\gamma_1^{-1}(P_2)$, a point in $\mathcal{U}$



Try to Apply Φ.... Not defined since ∞-ly many pts in $\gamma_2^{-1}(P_2)$.

if $\varphi = \pi/2$, then at north pole regardless of φ not bijective.

unit sphere with north, south poles

$$(\theta, \varphi) \mapsto \begin{pmatrix} \cos\theta\cos\varphi \\ \sin\theta\cos\varphi \\ \sin\varphi \end{pmatrix}$$

End up taking

$$\mathcal{U}' = \mathcal{U} - (X_u \cup \gamma_1^{-1} \circ \gamma_2(X_v))$$

$$V' = V - (X_v \cup \gamma_2^{-1} \circ \gamma_1(X_u))$$

$\Phi : \mathcal{U}' \to V'$ is ok.

Examples that are fully computable are rare. Often get functions under square root with no elementary antiderivative - have to resort to numerical integration. Simple example: surface $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$

$$D\gamma(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial}{\partial x} f & \frac{\partial}{\partial y} f \end{pmatrix}$$

$$D\gamma^T D\gamma = \begin{pmatrix} 1 & 0 & \frac{\partial}{\partial x} f \\ 0 & 1 & \frac{\partial}{\partial y} f \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial}{\partial x} f & \frac{\partial}{\partial y} f \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{\partial}{\partial x} f^2 & \frac{\partial}{\partial x} f \frac{\partial}{\partial y} f \\ \frac{\partial}{\partial y} f \frac{\partial}{\partial x} f & 1 + \frac{\partial}{\partial y} f^2 \end{pmatrix}$$

As surface in $\mathbb{R}^3$,

so parametrization $\gamma : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

then $\det(D\gamma^T D\gamma)^{1/2}$

can be rewritten as

$$\left| \frac{\partial}{\partial x}\gamma \times \frac{\partial}{\partial y}\gamma \right|$$

cross product

produces vector normal to the plane spanned by $\frac{\partial}{\partial x}\gamma, \frac{\partial}{\partial y}\gamma$

whose area = area of //-ogram bounded by these vectors.

x, y: param. vars.

(or any other names for param. vars. you might want to consider...)

taking determinant: Left with

$$1 + \frac{\partial}{\partial x} f^2 + \frac{\partial}{\partial y} f^2$$

take square root.

$$\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

not likely to have simple antiderivatives.

table of integrals only has quadratic functions under radical

i.e. $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ need to be linear.