Differentiating under integral sign:

Define $F(t) = \int_{\mathbb{R}^n} f(t, \underline{x}) \, |d^n \underline{x}|$ $f(t, \underline{x}) : \mathbb{R}^{n+1} \to \mathbb{R}$

integrable for any fixed t.

If $\frac{\partial}{\partial t} f(t, \underline{x})$ exists a.e. $\underline{x}$ and

$\exists$ $\delta > 0$ ~~then~~, integrable g s.t. (for all t_1, t_2) if $|t_1 - t_2| < \delta$ then

$$\left| \frac{f(t_1, \underline{x}) - f(t_2, \underline{x})}{t_1 - t_2} \right| \leq g(x)$$

then F is diff. in t with

derivative $$\frac{d}{dt} F(t) = \int_{\mathbb{R}^n} \frac{\partial}{\partial t} f(t, \underline{x}) \, |d^n \underline{x}|.$$

this gives us chance to use generalized Dominated conv. thm in proof. (*)

Do simple example.

If compact region, then (*) easily satisfied

if $\frac{\partial}{\partial t} f(t, x)$ is continuous.

Fourier transform: $$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) \, e^{i \xi x} \, dx$$

$i = \sqrt{-1}$.

$$e^{i \xi x} = \sum_{k=0}^{\infty} (i \xi x)^k / k! = \cos \xi x + i \sin \xi x.$$

If you prefer just think of writing function $g = g_1 + i g_2$ with g_1, g_2 real

Analyze behavior of Fourier transform under differentiation.

Possible example/non-example: e^{-*}, e^{*} ~~ξ~~

If we can differentiate under integral sign, then

$$\frac{d}{d\xi}\hat{f}(\xi) = \int_{\mathbb{R}} \frac{\partial}{\partial \xi}\underbrace{\left(f(x)e^{ix\xi}\right)}_{= ix\cdot f(x)e^{ix\xi}} dx = \widehat{ixf}(\xi)$$

(see this from power series or Euler's identity)

check this is possible exactly when $x \mapsto |xf(x)|$ is integrable. (*)

<u>Moral</u>: $\hat{f}$ differentiable when $|xf(x)|$ integrable (statement about decay of f at ∞. Can't go to ∞ like $1/x$ for example.)

So fourier transform: behavior at ∞ of f $\longleftrightarrow$ smoothness (i.e. diff.) of $\hat{f}$.

Also analyze $\widehat{f'}$ by parts, get

$$\widehat{f'}(\xi) = -i\xi\hat{f}(\xi)$$

so fourier transform turns differentiation into multiplication.

(*) <u>check</u>: $\left| \underbrace{\frac{e^{i(\xi+h)x} - e^{i\xi x}}{h}}_{\text{need to bound this for small } h} \cdot f(x) \right|$ bounded by function in x.

difference quotient in ξ var

$$= \left| \underbrace{e^{i\xi x}}_{\text{size } 1} \cdot \frac{e^{ihx}-1}{h} \right|$$

$$e^{ihx} = \underbrace{\cos hx}_{1-(hx)^2+\cdots} + i\underbrace{\sin hx}_{ihx+\cdots}$$

$1-(hx)^2+\cdots$ $\to 0$ as $h \to 0$

$\frac{ihx+\cdots}{h}$ $\to ix$ with size $|x|$ as $h \to 0$.

We're diving back into manifolds. What are manifolds?

our notion of smooth manifold — locally graph of C^1 function
in $\mathbb{R}^n$

(k-manifold : $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$)
k free vars, $n-k$ dependent vars.

$\updownarrow$

locally zero locus of some $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$
with $DF(\underline{z})$ onto at every point.

(key condition of implicit function theorem)

Less time on : parametrizations of k-manifolds.

$$\gamma : \mathcal{U} \subset \mathbb{R}^k \to M$$

(e.g. parametriz curves attempt to parametrize one-manifolds)

$\uparrow$
worry about self intersections etc.

s.t. ① $\mathcal{U}$ open

② γ is C^1, one-one, onto M

③ $[D\gamma(\underline{u})]$ is one-one $\forall\ \underline{u} \in \mathcal{U}$.

(condition ① prevents $\gamma : (0, 2\pi] \to \mathbb{R}^2$, $t \mapsto \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ from being valid param. of unit circle.)

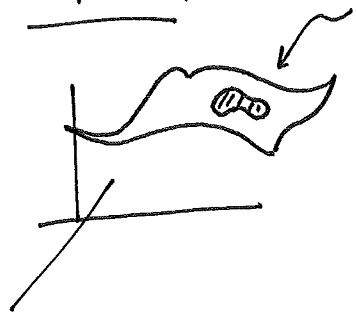
Very hard to give parametrizations.

(two sides of non-linear transformation
- non-linear kernel is zero locus
- non-linear image is parametrization)

But this is exactly what we need in order to define integral on manifold.

(See later that issues of circle parametrization ok since failure is on a set of measure 0)

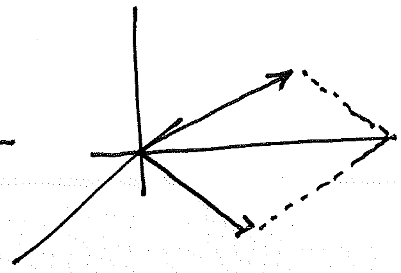
Picture of manifold



What is volume of  as subset of 2-manifold?

First small step in this direction — volumes of parallelograms.

Example: In $\mathbb{R}^3$,



two vectors define parellelogram $\underline{u}, \underline{v}$ in the plane containing $\underline{u}, \underline{v}$. What is its volume? (as 2-dim. manifold)

In Ch. 4, we learned volume of k-parallelogram in $\mathbb{R}^k$ spanned by $\underline{v}_1, \dots, \underline{v}_k$ is $|\det(\underline{v}_1, \dots, \underline{v}_k)|$

Clearly that doesn't work here: $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

gives 2×3 matrix.

clever fix:
$$\mathrm{vol}_k P(v_1, \dots, v_k) = |\det(v_1, \dots, v_k)|$$
$$= \sqrt{\det(T^T T)} \qquad T = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix}$$

(remember $\det(T^T T) = \det(T^T)\det(T)$
$= \det(T)^2.$)

this definition makes sense regardless of whether T square itself

if $T: m \times n$
$T^T: n \times m$
so
$T^T T$ is $n \times n$ matrix.

~~...~~ What does multiplication look like? If we have k vectors in $\mathbb{R}^n$.

$$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix}$$

$$\begin{bmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_k^T & - \end{bmatrix} \begin{bmatrix} |v_1|^2 & v_1 \cdot v_2 & \cdots \\ v_2 \cdot v_1 & \ddots & \vdots \\ \vdots & & \\ v_k \cdot v_1 & \cdots & |v_k|^2 \end{bmatrix}$$

in terms of $|v_i|, |v_j|$ and cosines of angles between them in k-plane.

(so independent of location of vectors in space - "anchor" as the book puts it.)

Book calls it definition. Not a definition.

Show it agrees with ~~...~~ volume in k-space if we take $v_1, \dots, v_k$ as basis for $\mathbb{R}^k \subseteq \mathbb{R}^n$.

<u>Plan</u>: Given suitable parametrization of k-manifold in $\mathbb{R}^n$ $\quad \gamma : \mathcal{U} \subset \mathbb{R}^k \longrightarrow M \subset \mathbb{R}^n$

pave $\mathcal{U}$ with cubes $\longmapsto$ weird shapes

But linearizing $D\gamma$: k-cubes $\longmapsto$ k-parallelograms

use these to approximate volume in M.

First, need to relax our notion of parametrization. Pre-conditions:

(*) boundary of $\mathcal{U}$ is of <u>k-dim'l volume</u> 0

Then $\gamma : \mathcal{U} \to M$ is a parametrization if

(1) $\gamma(\mathcal{U}) \supset M$

(2) $\exists\ X \subset \mathcal{U}$ with k-dim'l volume 0 s.t.

$\gamma(\mathcal{U} - X) \subset M$

(3) $\gamma : \mathcal{U} - X \to M$ is one-one, C^1 function with locally Lipschitz derivative

(4) $D\gamma(\underline{u})$ is one-one $\forall\ \underline{u}$ in $\mathcal{U} - X$.

(5) $\gamma(X) \cap C$ has k-vol $= 0$ $\forall$ compact $C \subset M$.