On Friday, learned that uniform convergence $\underset{\text{integrable}}{f_k} \to f$ is great, but

1 best: Dominated convergence theorem — Pick R with

$|f_k(\underline{x})| \leq R \quad \forall\, \underline{x}, \forall\, k,$

$\operatorname{Supp} f_k(\underline{x}) \subseteq B_R(0)$

then if $\lim_{k \to \infty} f_k$ converges to f integrable (up to set of measure 0)

$$\text{then} \quad \lim_{k \to \infty} \int_{\mathbb{R}^n} f_k(\underline{x})\, |d^n \underline{x}| = \int_{\mathbb{R}^n} \underbrace{\lim_{k \to \infty} f_k(\underline{x})}_{f}\, |d^n \underline{x}|$$

0 Proposed definition of "Lebesgue integral" as applied to infinite sum of integrable f_k:

$$\text{If} \quad \cancel{f = \sum_{k=1}^{\infty} f_k} \quad \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(\underline{x})|\, |d^n \underline{x}| < \infty \qquad (*)$$

then $\sum_{k=1}^{\infty} f_k$ converges to f almost everywhere. Define

$$\int_{\mathbb{R}^n} f(\underline{x})\, |d^n \underline{x}| \overset{\text{DEF}}{=} \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} f_k(\underline{x})\, |d^n \underline{x}|.$$

To make sure well-defined, if f_k, g_k satisfy $(*)$ and $\sum_k f_k = \sum_k g_k$

so converge to same function up to measure 0.

then their Lebesgue integrals are equal.

To prove it, we need dominated convergence thm.

0-th example: $f_1 = f$: Riemann integrable function, $f_2 = f_3 = \cdots = 0$.

$(f_k = 0 \text{ if } k > 1)$

then Lebesgue int. of $\sum_k f_k$ = Riemann int. of f.

1st example: $\displaystyle\int_{\mathbb{R}^n} \frac{1}{1 + |\underline{x}|^{n+1}} |d^n \underline{x}|$. If $n = 1$, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi$.

usual improper int. definition as $\displaystyle\lim_{A \to \infty} \int_{-A}^{A}$

Not Riemann integrable because, though output is always ≤ 1, support is not bounded.

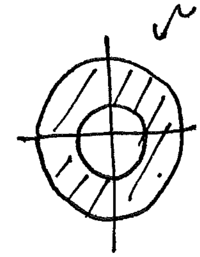
Need to use infinite series $\sum_k f_k$, with each f_k of bounded support.

E.g., since function is constant on sphere $|\underline{x}| = r$, break $\mathbb{R}^n$ into space between spheres of radius 2^k.

$$f_1 := \frac{1}{1 + |\underline{x}|^{n+1}} \cdot \mathbb{1}_{\text{unit ball}}$$

$$f_2 := \frac{1}{1 + |\underline{x}|^{n+1}} \cdot \mathbb{1}_{\text{ball of radius 2} - \text{ball of rad. 1}}$$

: in $\mathbb{R}^2$



a bounded set of support.

To show this is Lebesgue integrable, need to show:

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} f_k(\underline{x}) |d^n \underline{x}| < \infty.$$

Need to show: $$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} f_k(\underline{x})\, |d^n \underline{x}| = \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} g_k(\underline{x})\, |d^n \underline{x}|$$

Prove difference is 0. So show

$$\lim_{l \to \infty} \sum_{k=1}^{l} \int_{\mathbb{R}^n} \underbrace{(f_k - g_k)}_{\substack{\shortparallel \\ h_k}}(\underline{x})\, |d^n \underline{x}|$$ can be made arbitrarily small. (i.e. smaller than any $\epsilon > 0$.)

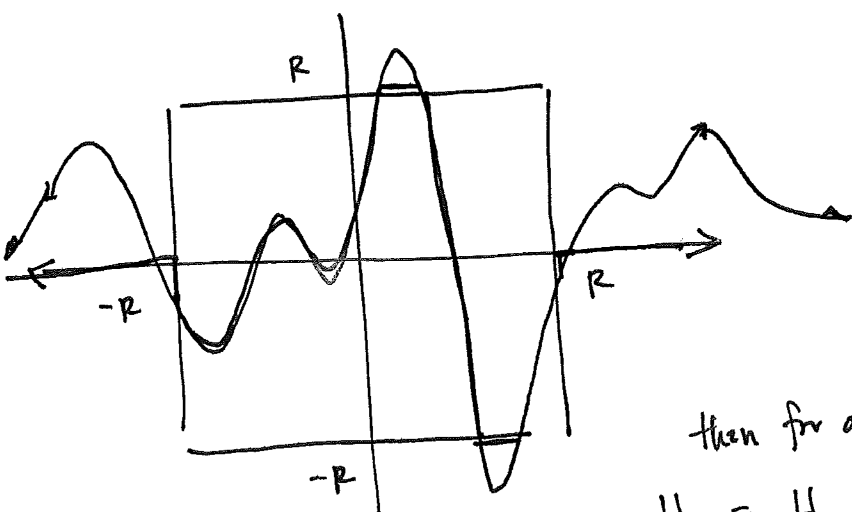
h_k (a function such that $\sum_{k=1}^{l} h_k =: H_l$ converges to 0 a.e. as $k \to \infty$.)

<u>Rewrite</u>: $$\boxed{\lim_{l \to \infty} \int_{\mathbb{R}^n} H_l(\underline{x})\, |d^n \underline{x}|.}$$

Done if we can bring limit inside. If H_l satisfies conditions of Dominated Convergence Theorem (i.e. we can find suitable R) then done.

Picture in $\mathbb{R}^1$: supported on $[-R, R]$ with values in $[-R, R]$

H_l's won't necessarily satisfy this, but truncated version does.



<u>Order</u>: Make tail $\sum_{k=M}^{\infty} \int |h_k|$ small, say $< \epsilon$ for sufficiently large M.

then pick R depending on $\underbrace{H_M}_{}$ finite sum of bounded functions. as h_k are Riemann integrable

then for any $l > M$:

$$H_l = \underbrace{H_l - H_l^{\text{trunc}(R)}} + \underbrace{H_l^{\text{trunc}(R)}}_{\substack{=0 \\ \text{by dom. conv.}}}$$

$$\underbrace{H_l - H_M}_{<\epsilon} \underbrace{- H_l^{\text{trunc}(R)} + H_M}_{\text{controllable}}$$

Clever idea: Map B_k : volume between sphere of radius 2^{k-1} and that of 2^{k-2}

f_1 is different, since unit ball topologically different from annuli.

Map all annuli to annulus between $R=1$ and $R=2$. $\Phi_k : \underline{x} \mapsto 2^{k-2}\underline{x}$ maps this annulus to support of f_k

$$\int_{\mathbb{R}^n} |f_k(\underline{x})| \, |d^n\underline{x}| = \int_{B_k} \frac{1}{1+|\underline{x}|^m} |d^n\underline{x}|$$

$$\underset{\substack{\uparrow \\ \text{change of vars.}}}{=} \int_{B_2} \underbrace{\left(\frac{1}{1+|\underline{x}|^m} \circ \Phi_k\right)}_{= \frac{1}{1+|2^{k-2}\underline{x}|^m}} (\underline{x}) \; \underbrace{\left|\det D\Phi_k(\underline{x})\right|}_{2^{(k-2)n}} |d^n\underline{x}|$$

$$\leq \int_{B_2} \underbrace{\frac{2^{(k-2)n}}{2^{(k-2)m}}}_{\text{need this sum over } k \text{ to converge.}} \frac{1}{|\underline{x}|^m} |d^n x|$$

Geometric series so need $m > n$.