topics so far in 3593:

Riemann integral
(defined by Riemann sum)

Theory:

1. Functional properties of integral
2. When is a function integrable?
3. (minor) other pavings (expanding definition?)

Computation:

1. Approximate methods:
   - weighting schemes (Gauss, Simpson)
   - Monte Carlo methods

Exact methods:

2. Fubini's theorem and changes of coordinates.

What to know about probability?

- definition of expected value / std. deviation
- statement of central limit theorem

two versions related by substitution:

$$P(\bar{x} \in [A,B]) = \frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma} \int_A^B e^{-\frac{n}{2}\left(\frac{x-E}{\sigma}\right)^2} dx$$

or

$$P\left(\bar{x} \in \left[E + \frac{\sigma}{\sqrt{n}}a\,,\, E + \frac{\sigma}{\sqrt{n}}b\right]\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-y^2/2}\,dy$$

average over n trials.

What about other pavings?

- we can generalize definition of Riemann int. to include other pavings.
- know definition of paving, nested partition, be able to check if given collection of sets satisfies definition.

What to know about measure 0?

- definition of measure 0
- used in best statement of integrability (iff)
- examples of sets of meas 0, not volume 0.

Example in theory: If f, g integrable. Prove $f+g$ integrable

and $$\int_{\mathbb{R}^n} (f+g)\,|d^n\underline{x}| = \int_{\mathbb{R}^n} f\,|d^n\underline{x}| + \int_{\mathbb{R}^n} g\,|d^n\underline{x}|.$$

proof: $$\int_{\mathbb{R}^n} f\,|d^n\underline{x}| = \lim_{N\to\infty} U_N(f) = \lim_{N\to\infty} L_N(f).$$

Compare: $U_N(f+g)$ vs. $U_N(f) + U_N(g)$.

$L_N(f+g)$ vs. $L_N(f) + L_N(g)$

Proofs quiz for students:

(A) if $f \le g$, f, g integrable, prove $\int_{\mathbb{R}^n} f\,|d^n\underline{x}| \le \int_{\mathbb{R}^n} g\,|d^n\underline{x}|$

(B) if $f, g : \mathbb{R}^n \to \mathbb{R}$ integrable, is $f\cdot g$ integrable?

(pf. or counterexample)

When $f\cdot g$ integrable, is it true that

$$\int f\,|d^n\underline{x}| \cdot \int g\,|d^n\underline{x}| = \int f\cdot g\,|d^n\underline{x}| ?$$

(proof or counterexample)

(C) Prove that volume is invariant under translation.

A pavable. $\vec{v} \in \mathbb{R}^n$. Show $\mathrm{vol}_n(A+\vec{v}) = \mathrm{vol}_n(A)$.

Important results: Prop 4.1.14, 4.1.16, 4.1.22, 4.1.24, Cor 4.3.8 graph vol 0. Cor 4.4.8. equal inf

Thm 4.9.4: volume/dets

(B) - part 1 is true. Use condition that f integrable $\iff$ f is discontinuous on set of at most measure 0.

this is still measure 0 on unions.

(even countably infinite unions, but we only need 2)

- part 2 is false.

Fails in one variable for most any pair of non-const. functions.

(C) Need to use definition here. Understand that lower sum: count cubes entirely in A

upper sum: count cubes that have non-empty intersection with A.

$$L_N(1_{A+\underline{v}})$$

$$= \sum_{C_i^N : C_i^N \subseteq A+\underline{v}} \text{vol}(C_i^N)$$

cubes are disjoint

$$= \sum_{D_i^N : D_i^N \subseteq A} \text{vol}(D_i^N + \underline{v})$$

formal change of vars.

$$= \sum_{D_i^N : D_i^N \subseteq A} \text{vol}(D_i^N)$$

$$= L_N(1_A)$$

Issue with this proof:

D_i^N's aren't dyadic cubes if C_i^N's are dyadic cubes.

This is why book instead uses:

$$1_{\substack{\cup C_i \\ \text{s.t. } C_i+\underline{v} \subseteq A}} \leq 1_{(A+\vec{v})} \leq 1_{\substack{\cup C_i \\ \text{s.t. } C_i+\underline{v} \\ \cap A \neq \phi}}$$

ensure these are dyadic cubes.