Last week, studied determinants algebraically.

Proved lots of properties using fact that they behave well under column operations. One more fact:

FACT 1: A is invertible $\Leftrightarrow$ $\det(A) \neq 0$.

pf: $\det(A) \neq 0$ $\Leftrightarrow$ A column reduces to identity

$\Leftrightarrow$ A is invertible (one-one, onto)

same pf as A row reduces to identity $\Leftrightarrow$ A invertible.

or $\Leftrightarrow$ A^T row reduces to identity

$\Leftrightarrow$ A^T invertible

$\Leftrightarrow$ A invertible.

$(A^T B) = I \Leftrightarrow (B^T \cdot A) = I$.

Eigenvalues/vectors of matrix A:

$$A \underline{v} = \lambda \underline{v} \quad \text{some } \lambda \in \mathbb{R}, \ \underline{v} \in \mathbb{R}^n$$

eigenvalue eigenvector.

vectors which get simply scaled under A.

How to find them:

Write identity $(A - \lambda I_n) \cdot \underline{v} = 0$.

Want $\underline{v} \in \ker(A - \lambda I_n)$ for some λ. Pick λ so that $A - \lambda I_n$ not one-one.

i.e. such that $\det(A - \lambda I_n) = 0$.

polynomial in λ (deg n), "characteristic polynomial", and so we solve for its roots in λ.

Find these roots $\lambda_1, \dots, \lambda_n$ and corresponding vectors $v_1, \dots, v_n$.

Make $v_1, \dots, v_n$ new basis for tranformation. Do example.

Find the eigenvalues/vectors for the transformation

$$T = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}.$$

To do this we compute $\det(T - \lambda I_3) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 3 & 1-\lambda \end{vmatrix}$

viewing λ as a variable,

solving for roots.

$$= 1-\lambda \cdot (1-\lambda)^2 - 0 + 1 \cdot (0 - (1-\lambda))$$

$$= (1-\lambda)\left(\underbrace{(1-\lambda)^2 - 1}_{\lambda^2 - 2\lambda}\right) = (1-\lambda)\lambda(\lambda - 2)$$

so $\lambda = 0, 1, 2$ are eigenvalues.

Now find elts in the $\ker(T - \lambda I_3)$ for each such choice of λ.

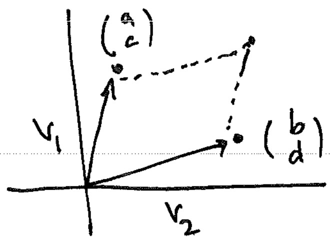
(Note: guaranteed to find basis of eigenvectors since we need three lin. ind. vectors, have 3 distinct e-values with non-trivial kernels)

e.g. $\ker(T - 1 \cdot I_3) = \ker \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 3 & 0 \end{pmatrix} \to \ker \begin{pmatrix} 1 & 0 & -3/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$

so pick $\underline{v} = \begin{bmatrix} 3/2 \\ -1/2 \\ 1 \end{bmatrix}$

or if we don't like fractions, use $2\underline{v}$

Volume using determinants:



then $\left| \det(v_1, v_2) \right|$ = area of parallelogram.

2×2 matrix with column vectors $\vec{v}_1, \vec{v}_2$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Or, thinking of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = T$ as linear transformation, then

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\text{unit cube}) =$ parallelogram above. Since unit cube has vectors $\binom{1}{0}, \binom{0}{1}$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{1}{0} = \binom{a}{c}$

— " — $\binom{0}{1} = \binom{b}{d}$

as a linear transformation,

$$\text{vol}(T(\text{unit cube})) = |\det(T)| \cdot \text{vol}(\text{unit cube})$$

Thm: If A is pavable set in $\mathbb{R}^n$, then write $T: \mathbb{R}^n \to \mathbb{R}^n$ linear transformation, $T(A)$ is pavable, and

$$\text{vol}_n(T(A)) = |\det(T)| \cdot \text{vol}_n(A).$$

Cool example: circle $x^2+y^2=1 \longmapsto$ ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

under map $T: \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

$$\Rightarrow \text{vol}(\text{ellipse}) = \text{vol}(\text{circle}) \cdot \det(T) = \pi ab.$$

pf of theorem: easy bit - if T not invertible, then $\det(T) = 0$

so must show $\mathrm{vol}(T(A)) = 0$ for any pavable A. But if

A pavable, then ~~A~~ bounded so $T(A)$ bounded. T has image in $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ since not invertible.

so $\mathrm{vol}_{\underline{\underline{n}}}(T(A)) = 0$ as bounded set in $\mathbb{R}^{n-1}$.

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hard bit - if T invertible, then the $\{T(C)\}$, C: Dyadic cubes of level N,

form a paving of $\mathbb{R}^n$. (easy to check)

now book gives 4 point plan of attack:

1. Show $\{T(D_N)\}$ give nested partition

2. Show property is true for unit cubes of dim. n, Q_n.

$$\mathrm{vol}_n T(Q_n) = |\det(T)| \, \mathrm{vol}_n Q_n = |\det(T)|$$

3. Show that have desired rel'n on dyadic cubes

$$\mathrm{vol}_n T(C_i^N) = \underbrace{|\det(T)|}_{} \cdot \mathrm{vol}_n(C_i^N)$$

Better to leave this as $\mathrm{vol}_n T(Q_n)$ ✓

4. Finally show true for pavable sets. A.

For 1, need that pieces are "nested" if $C_1 \subset C_2$ then $T(C_1) \subset T(C_2)$.

and that $\underbrace{\text{size}}_{\text{diam. of } A} \to 0$ as $N \to \infty$. True since $\|Tv\| \le |T| \cdot \|v\|$

so $\mathrm{diam}(TC) \le |T| \, \mathrm{diam}(C)$

Given 2, 3 is easy since $\dfrac{\mathrm{vol}_n T(C)}{\mathrm{vol}_n T(Q_n)} = \dfrac{\mathrm{vol}_n(C)}{\mathrm{vol}_n(Q_n)} = \mathrm{vol}_n(C)$

4. use definition of upper/lower Riemann sums as always. Not so hard, given 3.

Finally, to prove ②, we use fact that det is multiplicative, so

If $\mathrm{vol}_n(T(S(Q_n))) = \mathrm{vol}_n(T(Q_n))\,\mathrm{vol}_n(S(Q_n))$, (*)

then since both sides of desired identity behave well under mult. of matrices we can check the property on elementary matrices, which is easy.

Why is (*) true? Apply ④ with $A = S(Q_n)$. //