Determinants: Give a recursive definition or axiomatic definition.

(determinants give volume expansion by action of linear map)

Recursive: Illustrate first column expansion recursive definition.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \text{then } \det(A) = 1 \cdot \det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 4 \cdot \det\begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} + 7 \det\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$$

take det. of submatrix of rows, columns away from $a_{i,1}$

signs alternate

So picture $1 \cdot \det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$ as $\begin{pmatrix} 1 & \cancel{2} & \cancel{3} \\ \cancel{4} & 5 & 6 \\ \cancel{7} & 8 & 9 \end{pmatrix}$

You can take as definition the expansion along any fixed column. Not yet clear that they all produce same #. (be careful with signs)

Book uses symbol Δ_n for det. of $n \times n$ matrix.

$$\Delta_1([a]) = a. \qquad \Delta_n(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i,1} \cdot \Delta_{n-1}(A_{i,1})$$

$n-1 \times n-1$ matrix made from deleting row i and col. 1.

Thm: Δ_n is the unique function $f : \underbrace{(\mathbb{R}^n)^n}_{} \to \mathbb{R}$

n-tuples of vectors in $\mathbb{R}^n$ (i.e. $n \times n$ matrix)

s.t.

1. f is linear in all components
2. f is antisymmetric
3. $f(I_n) = 1$. (Better: $f(e_1, \ldots, e_n) = 1$.)

to prove that Δ_n satisfies these properties, prove by induction.

(need to break into cases according to whether column involved

is our expansion column or not.)

to prove that Δ_n is the unique such function, we use the axioms to prove how matrices behave under column operations:

(in fact, maybe you noticed that column ops are closely related to axioms)

① multiply column by constant. Multi-linearity implies that $\Delta_n(A_2) = m \cdot \Delta_n(A_1)$.

~~i.e. make new matrix~~

A_2 from A_1 by multiplying one column by m.

② Interchanging columns. Anti-symmetry implies

$$\Delta_n(A_2) = -\Delta_n(A_1).$$

③ Adding multiple of one column to another. Multilinearity implies:

$$\Delta_n(\underline{a}_1, \dots, \underline{a}_n) = \Delta_n(A) + \Delta_n(\underline{a}_1, \dots, \beta \underline{a}_k, \dots, \underline{a}_n)$$

$\underline{a}_j + \beta \underline{a}_k$ in j^{th} pos.

j^{th} position

so $\underline{a}_k$ appears twice here.

Prove in HW that this makes det $= 0$.

(Note: Proof needs to use axioms, not expansion definition, since we're using this fact to prove uniqueness of exp. def.)

so to prove uniqueness, do column operations to reduce to column echelon form. If one column has all 0's, then det (REF(A)) = 0

$\Rightarrow$ det (A) = 0. (since all operations reversible, and affect the det. by mult. by non-zero const.)

If REF(A) = I_n, then of course

det (REF(A)) = 1 $\longrightarrow$ working backward, just take inverse of constants in all column operations to get det (A).

this shows value of det(A) is determined by operations to get to REF(A). Hence unique if it exists.

Book has nice aside about computational advantages of using REF algorithm to compute det, versus factor expansion.

Laundry list of important properties of determinant.

FACT 1 : A is invertible $\iff$ $\det(A) \neq 0$

(pf: $\det(A) \neq 0 \iff$ column operations reduce to identity

Earlier proved A is invertible $\iff$ row operations reduce to identity

can use same proof to show true if column reduce to identity)

or wait a few more facts...

FACT 2 : A, B $n \times n$ matrices.

$$\det(A)\det(B) = \det(AB)$$

(clever pf: show $A \mapsto \det(AB)/\det(B)$ satisfies three characteristic properties of det, thus must equal the function $A \mapsto \det(A)$.)

FACT 3 : $\det(A^{-1}) = 1/\det(A)$ (if A invertible)

(pf: $\det(A \cdot A^{-1}) = \det(I) = 1$, now use fact 2.)

FACT 4 : $\det(P^{-1}AP) = \det(A)$ if P invertible

(pf: immediate from fact 3.)

FACT 5 : $\det(A) = \det(A^T)$.

(pf: show it is true for elementary matrices. $\leftarrow$ mult. by them corresponded to elem. row ops.

Examples :

scale	linear comb.	swap
$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 4 \end{bmatrix}$	$\begin{bmatrix} 1 & & 7 \\ & 1 & \\ & & 1 \end{bmatrix}$ det always 1	$\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}$ det is power of (-1)
$\det(E) = \det(E^T)$ since $E = E^T$	something small to check.	

now use fact that

$$\mathrm{REF}(A) =: \tilde{A} = \overbrace{E_k \cdots E_1}^{\text{sequence of mults. by elem. matrices}} A \qquad (*)$$

so taking transpose and remembering that $(AB)^T = B^T A^T$, then

$$\underbrace{\tilde{A}^T}_{\text{now in column echelon form}} = A^T E_1^T \cdots E_k^T . \qquad (**)$$

so taking determinants of both sides in each of $(*)$ and $(**)$

$$\det A = \det \tilde{A} \,/\, \det(E_k) \cdots \det(E_1) \qquad \det A^T = \det \tilde{A}^T \,/\, \det(E_1^T) \cdots \det(E_k^T)$$

so denoms equal, and for numerators,

know $\tilde{A} = I$ gives $\det(\tilde{A}) = 1 = \det(\tilde{A}^T)$

if $\tilde{A} \neq I$ has a row of 0's, and ~~dim~~ row space = dim column space

so $\det(\tilde{A}) = \det(\tilde{A}^T) = 0$.

e.g.
$$\begin{pmatrix} 1 & 4 & 0 \\ 0 & 0 & 1 \\ \boxed{0 \quad 0 \quad 0} \end{pmatrix}$$

FACT 6 : Determinant of triangular matrix is product of diagonal entries

(use induction)

same ideas can be used to calculate determinants of block matrices.

Other topics to mention :
1. Permutation formula for dets.
2. Characteristic poly. and eigenvalues
3. Trace. Invariance under change of coords.