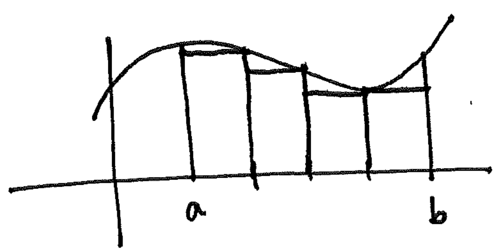
Integration theory. – in one-variable integration, definition of (definite) integral was via Riemann sum. In pictures, calculating area under curve by successively finer approximations:



Pick partition of $[a,b]$ into

$x_0 = a, x_1, \dots, x_n = b$

let # of parts $n \to \infty$ as their width gets smaller ($\to 0$ as $n \to \infty$)

usually width is regular so $\frac{b-a}{n}$.

Also have a choice of where to sample, sometimes test point notated $x_i^* \in [x_{i-1}, x_i]$

(left endpts / right endpts / midpts / mins / maxes)
mins " Lower sum, maxes " Upper sum.

Expectation: if f is nice enough, and partition width $\to 0$ as $n \to \infty$, all these sums converge (with different sampling rules) to same real number, this is value of integral.

Miracle of FTC: if f is really nice (elementary function), we can find this limit exactly using anti-derivative F of f. "indefinite integral"

One peculiarity of Riemann sums: signed area. So $\int_a^b f = -\int_b^a f$.

(in Riemann sum: $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,(\underbrace{x_i - x_{i-1}}_{\text{signed distance}})$. Could have put $|x_i - x_{i-1}|$ instead)

That issue becomes much more complicated in higher dimensions, so initially we're aiming for notion of absolute area/volume. Still could be negative depending on $f < 0$ or > 0.

("positive orientation")

Change in point of view:

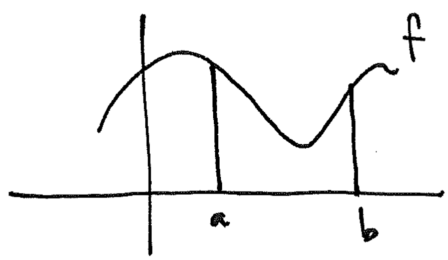
$$\int_a^b f \underset{\sim}{dx} = \int_{\mathbb{R}} g(x)\, dx$$

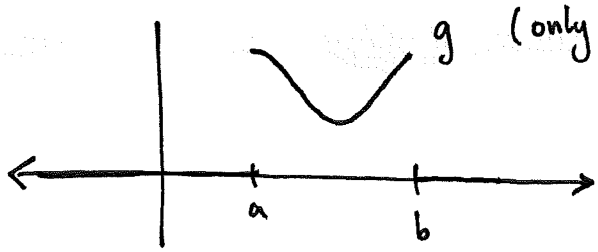
or $|dx|$

Denote this by $|dx|$ to distinguish from dx.

Slightly more verbose: Book uses $|d^n x|$ for integrals over $\mathbb{R}^n$.

where





(only non-zero on $[a,b]$)

Moved complexity of definition of domain into definition of g. In particular, g not continuous on $\mathbb{R}$. Leads to prettier definition.

With this in mind, make some initial assumptions about which functions we try to integrate: $f: \mathbb{R}^n \to \mathbb{R}$

1. Want $|f|$ to be bounded.
2. Want $\mathrm{Supp}(f) := \{ x \in \mathbb{R}^n \mid f(x) \neq 0 \}$ bounded.

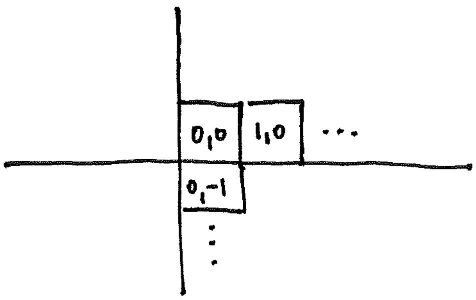
Remember, bounded sets are those contained in $B_r(0)$ for some r.

then, if integral is defined, then $\int_{\mathbb{R}^n} |f| \, |d^n x| < \infty$.

Finally, how to define Riemann integration for $\mathbb{R}^n$? Partition $\mathbb{R}^n$ into pieces. Always do same partition $\rightarrow$ cubes with side 1, cubes with side $1/2$, $\dots$, cubes with side $1/2^N$.

"dyadic paving"

Really particular – label them consistently using n-tuples of integers of ~~width~~ width $1/2^N$. So need 2^N cubes to get to $(1,0,\dots,0)$ starting at $(0,\dots,0)$



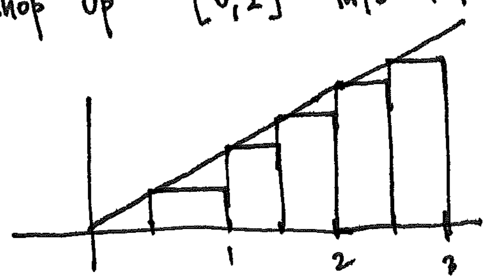
<u>Definition</u>: f is integrable if upper sum(f) = lower sum(f) using limit of dyadic pavings. (and then say $\int_{\mathbb{R}^n} f \, |d^n x|$ = upper sum(f) = lower sum(f))

<u>Example</u>: $\int_0^3 x \, dx$

Think of this as $\int_{\mathbb{R}} g(x) \, |dx|$ where $g(x) = \begin{cases} x & \text{if } x \in [0,3] \\ 0 & \text{else.} \end{cases}$

For each N, chop up $[0,2]$ into intervals ("1-cubes") of width $1/2^N$

$N = 1$:



lower sum: $\sum_{\text{intervals}} g\begin{pmatrix}\text{min in} \\ \text{interval}\end{pmatrix} \cdot \begin{pmatrix}\text{volume of} \\ \text{interval}\end{pmatrix}$

In dyadic paving, volume is always constant $= \left(\frac{1}{2^N}\right)^n$ n: dim, $N \to \infty$

lower sum: $\displaystyle\sum_{k=0}^{3\cdot 2^N-1} g\left(\frac{k}{2^N}\right)\cdot\frac{1}{2^N} = \frac{1}{2^N}\sum_{k=0}^{3\cdot 2^N-1}\frac{k}{2^N}$

similarly, upper sum:

$$\sum_{k=0}^{3\cdot 2^N-1} g\left(\begin{matrix}\text{max on}\\ \text{interval}\\ k\end{matrix}\right)\cdot\frac{1}{2^N} = \frac{1}{2^N}\sum_{k=0}^{3\cdot 2^N-1}\frac{k+1}{2^N}$$

We have $\displaystyle\sum_{k=0}^{3\cdot 2^N-1} k = \frac{(3\cdot 2^N-1)(3\cdot 2^N)}{2}$ (multiplied by $1/2^N\cdot 2^N$) in lower sum

so lower sum: $\displaystyle\lim_{N\to\infty}\frac{9}{2}\underbrace{\frac{(2^N-1/3)\cdot 2^N}{2^N\cdot 2^N}}_{\to 1 \text{ as } N\to\infty} = \frac{9}{2}.$

similarly upper sum: $\displaystyle\frac{3\cdot 2^N(3\cdot 2^N+1)}{2}\cdot\frac{1}{2^N\cdot 2^N} \to \frac{9}{2}$ as $N\to\infty$.

Immediate remark: If we can find criterion on g such that g is integrable (i.e. $\int_{\mathbb{R}^n} g\,|d^n x| =$ upper and/or lower sums)

then you can compute integral using dyadic paving and any sample points. This is just because, if x_k^* are sample points in interval k, (or cube)

then for all k $\quad g\left(\begin{matrix}\text{min on}\\ k^{th}\ \text{cube}\end{matrix}\right) \le g(x_k^*) \le g\left(\begin{matrix}\text{max on}\\ k^{th}\ \text{cube}\end{matrix}\right)$

so $\displaystyle L_N(g) \le \sum_{k \text{ in } C_N} g(x_k^*)\cdot\left(\frac{1}{2^N}\right)^n \le U_N(g)$

taking limits and using squeeze theorem shows this sampling for Riemann sum gives same #.