On Monday, discussed definition of limit of function:

$$\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = \underline{a} \quad \text{if for every } \epsilon > 0, \ \exists\ \delta(\epsilon) > 0 \text{ s.t.}$$

$$|\underline{x} - \underline{x}_0| < \delta \Rightarrow |f(\underline{x}) - \underline{a}| < \epsilon.$$

In particular, if $f: X \to \mathbb{R}^m$, $x_0 \in X$, say f is <u>continuous</u> at x_0 if $\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = f(\underline{x}_0)$. (and we say "continuous on X" if continuous for all $x_0 \in X$.)

Just as with limits of sequences, limits of functions converge if and only if each component function $f_i(\underline{x})$, $i = 1, \dots, m$, converges.

Again, get big list of properties for limits of functions by combining this result with our knowledge of one-var. limits.

H-H <u>Theorem 1.5.23 + Theorem</u> 1.5.24 limits of functions behave well under:

1. addition
2. "multiplication by scalar valued function" $h: \mathbb{R}^n \to \mathbb{R}$ $(\lim h \cdot f)$ vs. $(\lim h) \cdot (\lim f)$
3. dot product
4. composition (where both f, g defined)

Thm. 1.5.28 + 1.5.29 : same properties hold for continuous functions.

<u>cor</u> : polynomials, rational functions, combinations of "elementary functions" are continuous.

Proposition: $\mathcal{U}$: open in $\mathbb{R}^n$. $f: \mathcal{U} \to \mathbb{R}^m$ is continuous

if and only if the inverse image of every open set $V \subseteq \mathbb{R}^m$ is open.

denoted $f^{-1}(V)$

Interesting because didn't use the topology of open balls used to define open sets in $\mathbb{R}^n$. Could define open sets differently and get different family of continuous functions, taking above as the definition. Try to prove it!

($\Leftarrow$): Roughly, pick $B_\varepsilon(f(\underline{x}_0))$ it is open in $\mathbb{R}^m$ so inverse image is open and contains $\underline{x}_0$, so contains open ball around $\underline{x}_0$, take this to be δ-ball.

We could dive into derivatives now.

But we need one ingredient - Mean Value Thm

- to prove ~~that~~ a criterion for when a function is differentiable.

(among many other foundational results)

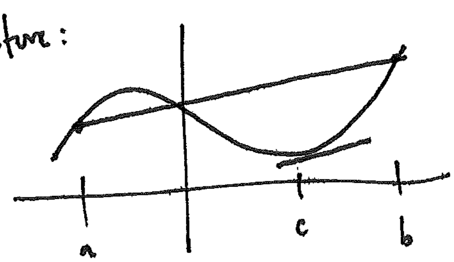
MVT: $f: [a,b] \to \mathbb{R}$ continuous, differentiable on (a,b)

then $\exists\, c \in (a,b)$ such that $\underbrace{f'(c)}_{\text{slope of tangent}} = \dfrac{f(b)-f(a)}{b-a}$.

slope of secant line

$\left(\text{Recall } f'(c) = \lim_{h\to 0} \dfrac{f(c+h)-f(c)}{h} \right)$

Common picture:



Plan: make function where desired c is maximum/minimum:

$h(x) := f(x) - \underbrace{(\text{secant line})}_{f(a) + \left[\frac{f(b)-f(a)}{b-a}\right](x-a)}$

Note $h(a) = h(b) = 0$ and continuous on $[a,b]$.

Now we need to show h has max or min. If $h \equiv 0$, then done.

If $h \not\equiv 0$, then it achieves either max or min (or both) somewhere in (a,b); call it c. Since f differentiable on (a,b), then h diff. on (a,b) so differentiable at c. Since c is max/min, then $h'(c) = 0$.

(remember $h'(c) = f'(c) - \left[\dfrac{f(b)-f(a)}{b-a} \right]$)

Two facts to prove: (1) if c is max/min of function $h : (a,b) \to \mathbb{R}$ differentiable then $h'(c) = 0$.

(2) $h : [a,b] \to \mathbb{R}$ continuous

then $\exists$ $c \in [a,b]$ with c : max i.e. $h(c) \geq h(x)$ for all $x \in [a,b]$

$\exists$ $c' \in [a,b]$ with c' : min i.e. $h(c') \leq h(x)$ $\forall$ $x \in [a,b]$

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(1) is easy. If c max, then $\displaystyle\lim_{H \to 0} \frac{h(c+H) - h(c)}{H} = \begin{cases} \leq 0 & \text{if } H > 0 \\ \geq 0 & \text{if } H < 0 \end{cases}$

But if h diff. at c, then only possibility is that the limit $= 0$. / Same idea for mins

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(2) is more subtle. Prove more generally for any $f : C \to \mathbb{R}$ where $C \subseteq \mathbb{R}^n$ is a compact set.

This statement (2) will follow from Bolzano-Weierstrass thm.

We say a subset $X \subset \mathbb{R}^n$ is bounded if it is contained in a ball of finite radius centered at ... the origin, say.

Then $C \subseteq \mathbb{R}^n$ is said to be compact if it is closed and bounded.

(think of examples where one of these is satisfied, not the other)