Had a first look at algebra associated with $\mathbb{R}^n$, now begin to study analysis. — need measure of size/length to define "closeness"

hence notion of limits, continuity, etc.

inner product : (dot product)

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \stackrel{\text{def}}{=} x_1 y_1 + x_2 y_2 + \cdots + x_n y_n .$$

nice properties : ① respects linear combinations (in both components) $(\vec{u}_1 + \vec{u}_2) \cdot \vec{v} = \vec{u}_1 \cdot \vec{v} + \vec{u}_2 \cdot \vec{v}$

and $(c \cdot \vec{u}) \cdot \vec{v} = c \cdot (\vec{u} \cdot \vec{v})$

(and similarly in second component)

② symmetric : $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
(aka commutative)

③ "positive definite" $\vec{u} \cdot \vec{u} \geq 0$ and $\vec{u} \cdot \vec{u} = 0$ only if $\vec{u} = \vec{0}$.

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Given inner product, define length (or norm) by $|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}}$

$= \sqrt{x_1^2 + \cdots + x_n^2}$

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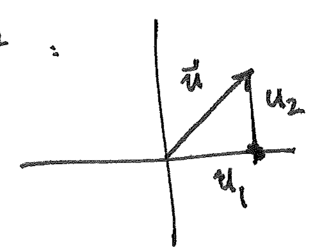
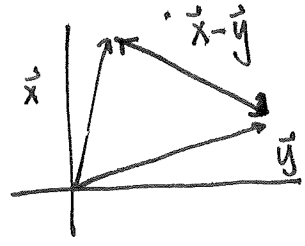
All this abstraction great, but comes from familiar calculations for length as in Pythagorean theorem. In $\mathbb{R}^2$:



then $|\vec{u}| = \sqrt{u_1^2 + u_2^2}$

—

Also consider triangle:
in $\mathbb{R}^2$



Two expressions for $|\vec{x} - \vec{y}|$

First: $|\vec{x}-\vec{y}|^2 = (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y}) \underset{\text{some algebra}}{= \cdots =} |\vec{x}|^2 + |\vec{y}|^2 - 2\vec{x}\cdot\vec{y}$

Second: "Law of cosines" $|\vec{x}-\vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 - 2|\vec{x}||\vec{y}|\cos\alpha$

where α: interior angle between $\vec{x},\vec{y}$.

comparing the two, we see $\vec{x}\cdot\vec{y} = |\vec{x}||\vec{y}|\cos\alpha$.

Upshot: Dot product is coordinate independent (RHS unchanged if we change coordinates)

Given two vectors in $\mathbb{R}^n$, if they aren't colinear then they span a plane and we can take their dot product, realizing answer as $|\vec{x}||\vec{y}|\cos\alpha$ where α is angle between $\vec{x},\vec{y}$ in plane. This proves that $\dfrac{\vec{x}\cdot\vec{y}}{|\vec{x}||\vec{y}|} \in [-1,1]$. (*)

This is the Cauchy-Schwarz inequality (incredibly useful ingredient in many proofs later):

Slightly rephrased: Given $\vec{v},\vec{w}$, $|\vec{v}\cdot\vec{w}| \le |\vec{v}||\vec{w}|$ with equality only if $\vec{v},\vec{w}$ colinear.

Book gives alternate proof valid for any inner product space. (and also makes their proof justification of (*), which is silly...)

Sketch of book proof: Expand $|t\vec{w}+\vec{v}|^2$, t: free variable, to get quadratic poly. in t whose discriminant is

$$4\left((\vec{v}\cdot\vec{w})^2 - |\vec{v}|^2|\vec{w}|^2\right) \qquad (**)$$

But $|t\vec{w}+\vec{v}|^2 \geq 0$, so disc. can't be positive, so $(**) \le 0$.

Corollary (triangle inequality) $|\vec{x}+\vec{y}| \leq |\vec{x}| + |\vec{y}| \quad \forall\, \vec{x},\vec{y} \in \mathbb{R}^n$

pf: $|\vec{x}+\vec{y}|^2 = |\vec{x}|^2 + 2\vec{x}\cdot\vec{y} + |\vec{y}|^2 \underset{\text{Cauchy-Schwarz}}{\leq} |\vec{x}|^2 + 2|\vec{x}|\cdot|\vec{y}| + |\vec{y}|^2$

$= (|\vec{x}| + |\vec{y}|)^2$

take square roots on both sides
since x^2 is monotonic function.

Can also define size on matrix A: $|A|^2 = \sum_{i,j} a_{ij}^2$ sum over all entries in matrix

only good if such a function is
compatible with basic operations.

So check (1) $|A\cdot\vec{b}| \leq |A|\cdot|\vec{b}|$

(2) $|A\cdot B| \leq |A||B|$

pf. sketch: if A is a row vector, then $A\cdot\vec{b}$ matrix mult.
is same as $A^T\cdot\vec{b}$ dot product ($A^T :=$ A written as column vector)
then follows from Cauchy-Schwarz.

For (1), Now think of A as bunch of row vectors
use Cauchy-Schwarz a bunch.

To prove (2), think of B as bunch of column vectors $\vec{b}$
and use (1) repeatedly.

Finally determinants. Define $\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc.$

What is motivation?

This appears in formula for inverse : $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

So in particular, 2×2 matrix is invertible if $ad-bc \neq 0$.

<u>Geometry</u>: $|ad-bc|$ is volume of parallelogram spanned by $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$

and sign of $ad-bc$ reveals orientation of $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$

travel clockwise / counterclockwise
$-$ / $+$

to get from $\begin{pmatrix} a \\ c \end{pmatrix}$ to $\begin{pmatrix} b \\ d \end{pmatrix}$ in smallest angle ($< \pi$)

We review determinant (and cross product)
more systematically later, but mention existence for now...

pf that $|\det(A)|$ is area of parallelogram spanned by column vectors in $\mathbb{R}^2$ is just a trigonometry problem, where we can extract the angle between vectors using dot product.

$\rightarrow$ let you read about cross-product, which exists only in $\mathbb{R}^3$
one or two HW questions using definition.