Last week – computing Taylor polynomials, for function $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$.

Two facts : (1) partial derivatives required are given by multi-indices

multi-indices remind us that mixed partials equal, remind us of factorial, exponent attached to monomial.

(2) shortcuts from composition / mult. rules for Taylor polynomials (power of little o / Big O notation)

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When we left off, working on Taylor polynomial to manifold at point : (using its implicit function thm)

$$M \longleftrightarrow F\begin{pmatrix} x \\ y \end{pmatrix} = x^3 + xy + y^3 - 3 = 0 \quad \text{at} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$DF\begin{pmatrix} 1 \\ 1 \end{pmatrix} = [4, 4] \Rightarrow \text{write } F = 0 \text{ locally as } \begin{pmatrix} \phi(y) \\ y \end{pmatrix}$$ (or in terms of x)

but hard to define ϕ explicitly in general, so can't compute Taylor poly. to ϕ directly.

Options : ① Harder : use higher derivatives + chain rule to find expressions for partial derivs of ϕ

② Better : use

$$P^k_{F, \begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix}} \begin{pmatrix} P^k_{\phi, \underline{b}}(\underline{b} + \underline{h}) \\ \underline{b} + \underline{h} \end{pmatrix} = 0$$ up to order k.

— return to example from last Wednesday

Final topic – Taylor's theorem with remainder, estimates difference between f, P_f.

In one variable

$$f(a+h) = f(a) + \int_0^h f'(a+t)\, dt \qquad \text{(fundamental thm. of calc.)}$$

Repeat using integration by parts:

$$f(a+h) = f(a) + f'(a)(h) + \cdots + \frac{1}{k!} f^{(k)}(a)(h)^k + \frac{1}{k!} \int_0^h (h - t)^k f^{(k+1)}(a+t)\, dt$$

Can't evaluate this latter integral, but we can estimate it using

Mean Value Thm: $\exists\ c \in (a, a+h)$ s.t.

$$\frac{1}{k!}\int_0^h (h-t)^k f^{(k+1)}(a+t)\,dt = \underbrace{\frac{1}{k!}\int_0^h (h-t)^k\,dt}\ f^{(k+1)}(c).$$

$t \mapsto h-t$

solve. get

$h^{k+1}/(k+1)!$

Don't know where c is exactly in $(a, a+h)$,

but if we bound $f^{(k+1)}(x)$ on this

interval (say $|f^{(k+1)}(x)| \leq C$, some const. C on this interval)

then

$$|\text{ERROR}| = \left|f(a+h) - P^k_{f,a}(a+h)\right| \leq \frac{C}{(k+1)!}h^{k+1}$$

Example: e^{-x^2} at $x=0$. $x \mapsto -x^2$, $0 \mapsto 0$ so can compose $x \mapsto -x^2$ with Taylor expansion for e^x at $x=0$

$$e^{-x^2} \approx 1 - x^2 + \frac{x^4}{2} + o(x^4)$$

What is bound for error of $e^{-1/4}$ using $P^2_{f,0} = 1 - x^2$?

We estimate at $x = 1/2$ that $e^{-1/4} \approx 3/4. = .75$

$$|\text{error}| \leq \frac{C}{3!}(1/2)^3 = \frac{C}{48}; \quad \text{where } |f^{(3)}(x)| \leq C \text{ on } (0, 1/2)$$

$$C = 4 \cdot 1/2 \cdot 3 \cdot 1 = 6$$

$f(x) = e^{-x^2}$

$f'(x) = -2x\,e^{-x^2}$

$f''(x) = 4x^2 e^{-x^2} - 2e^{-x^2}$

$f'''(x) = -8x^3 e^{-x^2} + 8x\,e^{-x^2} + 4x\,e^{-x^2} = -8x^3 e^{-x^2} + 12x\,e^{-x^2}$

$= -4x(x^2 - 3)e^{-x^2}$

A similar result exists for multi-var. functions: (clever application of one var. result

$\exists\ c \in [\underline{a}, \underline{a}+\underline{h}]$ s.t.

$$f(\underline{a}+\underline{h}) - P^{k}_{f,\underline{a}}(\underline{a}+\underline{h}) = \sum_{I \in \mathcal{I}_n^{k+1}} \frac{1}{I!} D_I f(\underline{c}) \cdot \underline{h}^I$$

So need to bound all these with $c \in [\underline{a}, \underline{a}+\underline{h}]$.

Say by constant C

then

$$|\text{Error}| \le C \cdot \left(\sum_{i=1}^{n} |h_i| \right)^{k+1} \qquad (\text{need } f \in C^{k+1})$$

Next topic: classifying local max/min of functions

In one variable, used second derivative test: if $f'(a)=0$ and ...

if $f''(a) > 0$: min, if $f''(a) < 0$: max.

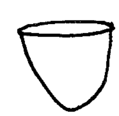
Recast: quadratic term in Taylor expansion is pos./neg.

What about functions $f: \mathbb{R}^2 \to \mathbb{R}$?
$(x,y) \mapsto f(x,y)$

Top terms in quadratic Taylor polynomial have form $a_{(2,0)} x^2 + a_{(1,1)} xy + a_{(0,2)} y^2$.

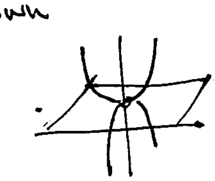
How to understand behavior of terms like this?

Prototypes: $f(x,y) = x^2 + y^2$ "paraboloid"



$f(x,y) = -x^2 - y^2$ same, but upside down

$f(x,y) = x^2 - y^2$ "saddle"



or $y^2 - x^2$