Last time, introduced Taylor polynomials $p^k_{f,\underline{a}}(\underline{x})$ to $f(\underline{x}) : U \to \mathbb{R}$.

Two facts: (1) $p^k_{f,\underline{a}}(\underline{x})$ is unique polynomial matching partial derivatives of f up to degree k.

(2) it is unique polynomial of deg $\le k$ such that

$$\lim_{\underline{h}\to 0} \frac{f(\underline{a}+\underline{h}) - p^k_{f,\underline{a}}(\underline{a}+\underline{h})}{|\underline{h}|^k} = 0.$$

Notation: $f(\underline{a}+\underline{h}) - p^k_{f,\underline{a}}(\underline{a}+\underline{h})$ is $o(|\underline{h}|^k)$

In general f is $o(g(x))$ if $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$.

"order" as in "on the order of"

really useful in thinking about taylor expansions.

e.g.: $f(x,y,z) = e^{(x+2y+z^2)}$ $\qquad e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + o(|x|^k)$

Find $p^2_{f,\underline{a}}(x,y,z)$ at $\underline{a} = (0,0,0)$

Long way: $D_1 f = e^{x+2y+z^2}\Big|_{0,0} = 1 \qquad D_3 f = 2z \cdot e^{x+2y+z^2}\Big|_{(0,0,0)} = 0$

$D_2 f = 2 \cdot e^{x+2y+z^2}\Big|_{0,0} = 2 \qquad D_3^2 f = 2e^{x+2y+z^2} + 4z^2 = 2.$

$D_1^2 f = e^{x+2y+z^2} = 1 \qquad \left.\begin{matrix} D_3 D_1 \\ D_3 D_2 \end{matrix}\right. = 0 \text{ @ } (0,0,0)$

$D_2 D_1 f = 2e^{x+2y+z^2} = 2$

$D_2^2 f\Big|_{(0,0,0)} = 4$

putting it together:

$$e^{x+2y+z^2} = 1 + (x-0) + 2(y-0) + 0(z-0)$$
$$+ \frac{1}{2!}\cdot(x-0)^2 + 2(x-0)(y-0)$$
$$+ \frac{4}{2!}(y-0)^2 + \frac{2}{2!}(z-0)^2$$

$$= 1 + x + 2y + \frac{x^2}{2} + 2xy + 2y^2 + z^2$$

Claim: same as substituting $x+2y+z^2$ into Taylor series for e^x

$$e^x = 1 + x + \frac{x^2}{2!} + O(|x|)^3.$$

$$e^{x+2y+z^2} = 1 + (x+2y+z^2) + \underbrace{\frac{(x+2y+z^2)^2}{2!}}_{\frac{x^2+2xy+z^2+4y^2}{2!} + \text{order 3 and higher terms}} + O(\underbrace{|x+2y+z|^3}_{\text{3rd degree and higher. won't appear in deg 2 Taylor poly.}})$$

Much faster.

General rule: Taylor polynomial of $f\circ g$ (both k times diff., compose nicely)

e.g. $g: \mathcal{U} \to V$, $f: V \to \mathbb{R}$, $\mathcal{U} \subset \mathbb{R}^n$, $V \subset \mathbb{R}$

$$P^k_{f\circ g,\underline{a}}(\underline{x}) = P^k_{f,\underline{b}}(P^k_{g,\underline{a}}(\underline{x}))$$ after discarding terms of order $> k$. (strict ineq.)

where $g(\underline{a}) = \underline{b}$

Lucky! in example $x+2y+z^2\big|_{(0,0,0)} = (0)$

Also sum and product rules.

$$P^k_{f+g,\underline{a}}(x) = P^k_{f,\underline{a}}(x) + P^k_{g,\underline{a}}(x)$$

$$P^k_{f\cdot g, a}(\underline{x}) = P^k_f(\underline{x}) \cdot P^k_g(\underline{x}) \quad \text{(after discarding terms of degree} > k)$$

If we work on smooth k-manifold M associated to zero locus $F = 0$.

then write $\underline{z}_0 = \begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix} \begin{matrix} \text{3 pivots} \\ \text{3 non-pivots} \end{matrix}$, evaluate $P^k_{\phi, \underline{b}}$ by

taking derivatives using implicit function thm.

Example: $F\begin{pmatrix} x \\ y \end{pmatrix} = x^3 + xy + y^3 - 3 = 0.$ $\quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is on zero locus.

At $(1,1)$, $DF(1,1) = \left[3x^2 + y \big|_{(1,1)},\ 3y^2 + x \big|_{(1,1)} \right] = [4, 4]$

either variable can be pivot var. Suppose we take x as pivot,

so $F = 0$ expressible as $\begin{pmatrix} \phi(y) \\ y \end{pmatrix}$ near $(1,1)$.

Then $\phi'(1) = -\left[DF_1(1,1)\right]^{-1} \cdot DF_2(1,1) = -\frac{1}{4} \cdot 4 = -1.$

Now what? Can we find $\phi''(1)$?

$F\begin{pmatrix} \phi(y) \\ y \end{pmatrix} = 0.$ chain rule:

$$0 = \frac{\partial F}{\partial y}\begin{pmatrix} \phi(y) \\ y \end{pmatrix} + \frac{\partial F}{\partial x}\begin{pmatrix} \phi(y) \\ y \end{pmatrix} \cdot D\phi(y)$$

So can take second derivatives here as well.

get exact expression.

Or use fact about chain rule of Taylor polynomials.

Get $P^k_{F, \begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix}} \begin{pmatrix} P^k_{\phi, \underline{b}}(\underline{b} + \underline{h}) \\ \underline{b} + \underline{h} \end{pmatrix} = 0$ up to order k.

So in our example, $P^2_{\phi,1}(1+h) = 1 - h + \frac{\phi''(1)\cdot h^2}{2!}$

Want to solve for $\phi''(1)$.

$$F\begin{pmatrix} P^2_{\phi,1}(1+h) \\ 1+h \end{pmatrix} = (1+h)^3 + (1+h)\,P^2_{\phi,1}(1+h) + P^2_{\phi,1}(1+h)^3 - 3$$

(could have first found $P^2_{F,(1,1)}$, but why bother because they agree up to quadratic factors)

$$\underset{\substack{\uparrow \\ \text{as element in } O(h^2)}}{=} 1 + 3h + 3h^2 + (1+h)\left(1-h+\frac{\phi''(1)h^2}{2}\right) + \left(1-h+\frac{\phi''(1)h^2}{2}\right)^3 - 3$$

We know $= 0$ in $O(h^2)$. Indeed check that constant and linear terms in $h = 0$.

What about quadratic term? $h^2\Big(\underbrace{3}_{\text{from } 3h^2} + \underbrace{\frac{\phi''(1)}{2} - 1}_{\text{from } (1+h)P^2_\phi} + \underbrace{3 + \frac{3\phi''(1)}{2}}_{\text{from } (P^2_\phi)^3}\Big) = 0$

so $\phi''(1) = \boxed{-\frac{5}{2}}$

Similarly solve for higher degree terms.

Proofs: Manipulate little o and Big O notation. (see Appendix K)

Say $h(x) > 0$ in nbd. of 0, then f is in $O(h(x))$ if $\exists\, \delta > 0$ and pos. const. C s.t. $|f(x)| \le C \cdot h(x)$ when $x \in B_\delta(0)$.