Last time, discussing smooth manifolds — hypersurfaces that locally are graph of differentiable (cont.) function. (i.e. C^1 fun.)

Thm 3.1.10 : Zero locus of $F : \mathbb{R}^{m+n} \to \mathbb{R}^m$, $F : C^1$, $DF(\underline{z})$ onto $\forall\, z$ in locus

implicitly defines smooth manifold :

$$M := \{ \underline{z} \in \mathcal{U} \mid F(\underline{z}) = \underline{0} \}$$ is smooth, n-dim'l manifold.

(pf: implicit function thm holds at all $\underline{z} \in M$)

Example: $x^4 + y^4 + x^2 - y^2 = c$. For which c does this map $\mathbb{R}^2 \to \mathbb{R}$ have assoc. zero locus defining smooth manifold? (1-dim'l) according to theorem

Last time: Find where derivative fails to be onto for $(x,y) \in \mathbb{R}^2$.

See if those points lie on our zero locus.

$$DF = [\, 4x^3 + 2x \,,\ 4y^3 - 2y \,]$$. Only place not onto if $DF = [0,0]$

$2x(2x^2+1)$ $\quad 2y(2y^2-1)$

only $=0$ if $x=0$ $\quad y = 0, \pm 1/\sqrt{2}$

if $x=0,\ y=0$. $\quad x^4 + y^4 + x^2 - y^2 = 0$. so $c=0$

$x=0,\ y=\pm 1/\sqrt{2}$ $\quad 0 + 1/4 + 0 - 1/2 = -1/4$ so $c = -1/4$

} can't guarantee manifold structure.

$c=0$ case : Figure 8 curve. Not manifold.

$c=-1/4$: two isolated points $(0, \pm 1/\sqrt{2})$. 0-dim'l manifold.

Converse to Thm 3.1.10: If M smooth, ~~___~~ n-dim'l manifold in $\mathbb{R}^{n+m}$

then every $\underline{z} \in M$ has nbhd $\mathcal{U} \subseteq \mathbb{R}^{n+m}$, C^1 function $F: \mathcal{U} \to \mathbb{R}^m$

s.t. $DF(\underline{z})$ onto and $M \cap \mathcal{U} = \{ \underline{y} \mid F(\underline{y}) = 0 \}$

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pf. sketch: Write $\underline{z} = \begin{bmatrix} \underline{y} \\ \underline{x} \end{bmatrix} \begin{array}{l} \} \text{ n indep.} \\ \} \text{ m dep.} \end{array}$. consider $F(\underline{z}) := \underline{x} - f(\underline{y})$

zero locus.

this is onto $\mathbb{R}^m$ since Jacobian is identity in columns corresponding to m x_i variables.

(here f is diff. function guaranteed by manifold definition.)

Thm: (Inverse image of manifold is manifold)

$M \subset \mathbb{R}^m$, k-dim'l manifold, with $\mathcal{U}$: open set in $\mathbb{R}^n$, $f: \mathcal{U} \to \mathbb{R}^m$ in $C^1(\mathcal{U})$

with Df onto for all $x \in f^{-1}(M)$. Then $f^{-1}(M)$ is a submanifold of $\mathbb{R}^n$ of dim'n $k+n-m$.

inverse image of f, a set.

Cor: ~~The~~ Let g be the map $\begin{array}{l} \mathbb{R}^n \to \mathbb{R}^n \\ \underline{x} \mapsto A \cdot \underline{x} + \underline{c} \end{array}$

with A invertible.

then if M smooth k-manifold,

$g(M)$ is smooth k-manifold.

pf of Cor: (assuming the theorem) $g^{-1} = f$ where $f: \underline{x} \mapsto A^{-1}(\underline{x} - \underline{c})$

and $g(M) = f^{-1}(M)$. Now apply thm. to f. (In short, didn't like direct images so used inverse image of inverse map!

really mean inverse!

In other words, notion of manifold is coordinate-free since all changes of coordinates are of form $\underline{x} \mapsto A\underline{x} + \underline{c}$.

<u>proof of theorem</u>: Given $\underline{a} \in f^{-1}(M)$, know that since $f(\underline{a}) \in M$ $\exists$ nbhd V of $f(\underline{a})$ s.t. points of $M \cap V$ are given by zero locus of

$F(\underline{y}) = 0$, $F: V \to \mathbb{R}^{m-k}$, C^1 mapping with DF onto for all $x \in M \cap V$.

Since f continuous $f^{-1}(V)$ open neighborhood of $\underline{a}$ and the set $f^{-1}(M) \cap f^{-1}(V)$ is the solns to $F \circ f = \underline{0}$.

Need to check $D(F \circ f)$ onto. Do this by chain rule. ✓ since $DF(f(\underline{y}))$ onto by part 2 of earlier thm, $Df(\underline{y})$ onto by assumption in thm.

Book goes on long discussion about parametrizations versus implicitly defined functions, by equation.

check a pt. is on equation's zero locus, but don't know how to find pts on locus in first place.

Just the opposite is true for ~~...~~ parametrizations.

Further Trouble with parametrizations: hard to explain why they have manifold structure, if indeed they do.

(Key point: mapping must be one-one.

e.g. $\mathbb{R} \to M$

$t \mapsto \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$

Locus of points traced out is Manifold, hard to prove from parametrization

Do in patches)

Tangent spaces : Recall tangent hyperplane to a function $f(\underline{x})$ at $\underline{x}_0$,

$f: \mathbb{R}^k \to \mathbb{R}^m$, is $T(\underline{x}) - f(\underline{x}_0) = [Df(\underline{x}_0)](\underline{x} - \underline{x}_0)$

with $Df(x_0) : \mathbb{R}^k \to \mathbb{R}^m$
linear transformation.

We can make the same definition for a smooth k-manifold. By defn, for any $\underline{z}_0 \in M \subset \mathbb{R}^n$, locally the graph of C^1 function. For $\underline{z} \in B_\varepsilon(\underline{z}_0)$,

$$\text{Write } \underline{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ y_1 \\ \vdots \\ y_{n-k} \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ f_1(x_1,\dots,x_k) \\ \vdots \\ f_{n-k}(x_1,\dots,x_k) \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \underline{x} \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \underline{f}(\underline{x}) \end{matrix}$$

Then

Note: reordered variables here to make it pretty.

tangent hyperplane is again $T(\underline{x}) - \underline{f}(x_0) = [Df(\underline{x}_0)](\underline{x} - \underline{x}_0)$.

Definition : The tangent space to manifold M at point $\underline{z}_0 \in M$, denoted $T_{z_0}(M)$, is the graph of the linear transformation $Df(\underline{x}_0)$ with f as above.

e.g. $y = f(x)$, then $Df(\underline{x}_0) = f'(\underline{x}_0)$

and graph is linear function $y = f'(\underline{x}_0) \cdot x$ (not tangent line. This we get by shifting $(0,0)$ to $(x_0, f(x_0))$)

Nicer from linear algebra point of view. Tangent space is vector space, but it is good to think about it as "anchored" to point $(\underline{x}_0, \underline{f}(\underline{x}_0)) \in \mathbb{R}^n$.