Even the most basic curves, surfaces, shapes may not be everywhere in their domain expressible as functions of given collection of independent variables.

Unit circle is expressible as function of one var. on half-circle, but not over whole circle. In some cases, we must be satisfied with expressing object <u>locally</u> as a function of some indep. variables.

"locally" means: in a neighborhood of each point.

"neighborhood" for x_0 means (open) set containing $B_\epsilon(x_0)$ for some $\epsilon > 0$.

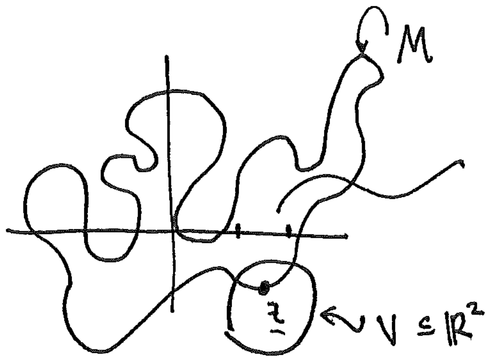
(By contrast "globally" refers to a property of function on entire domain. Met these terms in 1-var. calculus when discussing "local vs. global extrema")

Basic point: When a hyper-surface looks locally like a differentiable function, all the tools of calculus apply.

Motivates our definition of "manifold" (smooth) — Hubbard's definition is different than most everyone else. (more on this later)

Definition: A subset of $\mathbb{R}^n$, M, is a smooth k-dimensional manifold if, for every $\underline{z} \in M$, $\exists$ nbhd $V \subset \mathbb{R}^n$ such that $V \cap M$ is the graph of a C^1 function f of k variables.

Picture : $\mathbb{R}^2$. M : 1 dim'l manifold.



M looks like function of x on $V \cap M$

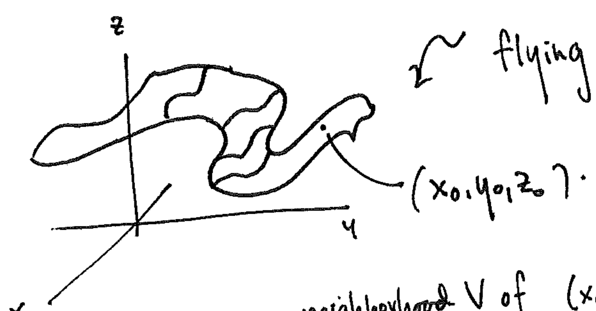
Means we can write points in $V \cap M$ as $\begin{bmatrix} x \\ \phi(x) \end{bmatrix}$ for some x.

How to check the unit circle is 1-dim'l manifold ?

(In this case, solve explicitly at any (x_0, y_0) s.t. $x^2 + y^2 - 1 = 0$)

(if this reminds you of implicit function thm, good since that will be a major method for finding manifolds)

Example 2 : Surfaces in $\mathbb{R}^3$: M : 2-dim'l manifold.



flying carpet bending back on itself.

(x_0, y_0, z_0).

neighborhood V of (x_0, y_0, z_0)

$V \cap M$

← nbhd. of (x_0, y_0) in x-y plane

Write points on $V \cap M$ as $\begin{bmatrix} x \\ y \\ \phi(x,y) \end{bmatrix}$

a C^1 function. on nbhd of (x_0, y_0) in xy-plane.

If we chose point on fold, might have to use nbhd. in xz-plane etc.

Easy example : Any (C^1) function $\phi(\underline{x}) : \mathbb{R}^k \to \mathbb{R}^m$ defines a k-dim'l manifold in $\mathbb{R}^{k+m}$: $\begin{bmatrix} \underline{x} \\ \phi(\underline{x}) \end{bmatrix}$

works for all points in $\mathbb{R}^{k+m}$. Only time we expect to get global def'n.

try this on own with unit sphere : $x^2+y^2+z^2-1=0$

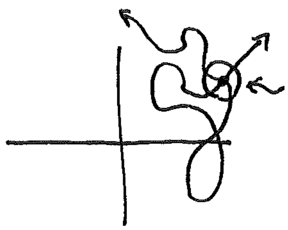
e.g. $\phi(x,y) \equiv 0 \; : \; \mathbb{R}^2 \to \mathbb{R}$. $\rightsquigarrow$ defines manifold structure on

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \subseteq \mathbb{R}^{2+1} = \mathbb{R}^3.$$

but this is just $\mathbb{R}^2$.

so $\mathbb{R}^2$ is manifold.

Non-examples: Self-intersecting curves in the plane:

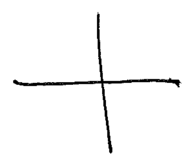


is  the graph of a function?

No, fails one-one in both directions.

Rigorous proof that any self-intersecting curve is not a manifold might be hard. But if we have equations, it is easier.

Simplest example: $xy = 0$.

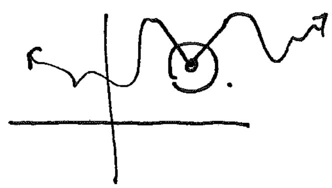


graph is coord. axes.

Fails one-one-ness badly at $(0,0)$.

If we remove origin, then we do have manifold.

Another non-example: sharp point in plane:



is locally  in nbhd of sharp point.

Not defined in open nbhd of y_0, not diff. in open nbhd of x_0.

Implicit function theorem gives us many more examples of smooth manifolds:

Thm 3.1.10 in Hubbards: $U \subseteq \mathbb{R}^n$ open, $F : U \to \mathbb{R}^{n-k}$ a C^1 mapping

Consider $M \stackrel{\text{def}}{=} \{ \underline{z} \in U \mid F(\underline{z}) = \underline{0} \}$ "zero locus of F"

If $DF(\underline{z})$ is onto $\forall\, \underline{z} \in M$, then M is a smooth k-dim'l manifold embedded in $\mathbb{R}^n$ (proof is just implicit function theorem)

There's a converse: If M is a smooth k-dim'l manifold embedded in $\mathbb{R}^n$, then every $\underline{z} \in M$ has nbhd $\mathcal{U} \subseteq \mathbb{R}^n$ s.t. $\exists$ C^1 $F: \mathcal{U} \to \mathbb{R}^{n-k}$ with $DF(\underline{z})$ onto and $M \cap \mathcal{U} = \{ \underline{y} \mid F(\underline{y}) = \underline{0} \}$.

— write $\underline{z} = \begin{bmatrix} \underline{y} \\ \underline{x} \end{bmatrix} \begin{matrix} \} k \text{ indep.} \\ \} n-k \text{ dep.} \end{matrix}$ $\xrightarrow{\text{consider}}$ $F(\underline{z}) := \underline{x} - f(\underline{y}) = \underline{0}$ (prove this Wednesday)

Nice remark in book: If $DF(\underline{z})$ fails to be onto, doesn't mean that zero locus isn't smooth manifold. (E.g. dirty trick $F(\underline{z})^2 = 0$

then $DF^2 = 2F\,DF \equiv 0$)

So even if $F(\underline{z}) = 0$ defines manifold then $F(\underline{z})^2 = 0$ (which is the same set) won't satisfy condition

—

Example in book: $x^4 + y^4 + x^2 - y^2 = c$.

For which c does this define smooth manifold? $c = -1/4$, $c = 0$ fail to be onto

$c = -1/4$: two points. Is this a smooth manifold? Yes. 0-dimensional.

$(0, \pm 1/\sqrt{2})$

$c = 0$: figure eight curve.

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k-dim'l Manifold in $\mathbb{R}^n$ $\longleftrightarrow$ local implicitly defined functions $\begin{pmatrix} \underline{x} \\ \phi(\underline{x}) \end{pmatrix}$

$\underline{x}$: subset of k-variables in $\mathbb{R}^n$

(reordered them here so they occur first in vector notation)