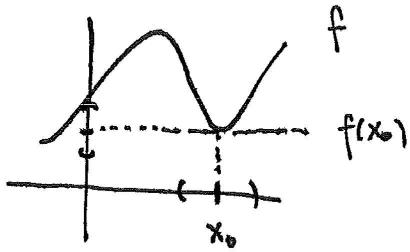
Friday, we discussed statements of inverse function theorem.

Qualitative: If $f: \mathcal{U} \to \mathbb{R}^n$ contin. diff., $x_0 \in \mathcal{U}$ with $Df(x_0)$ invertible, then f is invertible with diff. inverse in a nbhd. of $f(\underline{x}_0)$

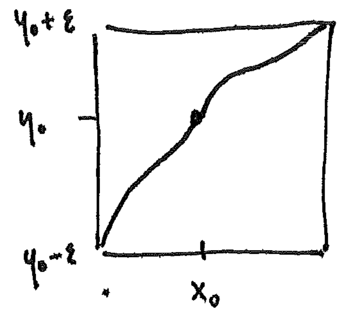
That is, the equation $f(\underline{x}) = y$ is solvable in $\underline{x}$ for all $y \in B_r(f(\underline{x}_0))$

Look at 1-variable pictures: $f: \mathbb{R}^1 \to \mathbb{R}^1$



Inverse function thm. fails since $f'(x_0) = 0$ not invertible.
Indeed, we can't solve for $f(x) = y$ if $y < y_0$.

Away from max/min:



Really do have solutions on $B_\varepsilon(y_0)$.

Another point of view: f satisfying above conditions (invertible derivative) is behaving like a linear system, namely the system $[Df(\underline{x}_0)]$, in nbhd. of $\underline{x}_0$.

Quantitative version: Don't want to restate this. Just remind you that we place extra condition that Df Lipschitz (stronger than Df continuous) with just right Lipschitz ratio to apply Kantorovich's Thm to

$$f(\underline{x}) - y = 0 \quad \text{for any } y \in B_R(y_0) \quad \text{with initial guess } x_0.$$

Remarks: ① Conditions of theorem show $x_0, x_1, \ldots$ converges to root of $f(\underline{x}) - y = 0$ so its limit is $f^{-1}(y)$.

① cont. - Still need to show f^{-1} differentiable with continuous partials.

(prove some inequalities - heart of analysis in these arguments. See pfs. in appendix)

② If $Df(\underline{x})$ invertible for all $\underline{x} \in \mathcal{U}$, still might not have global inverse.

that is, intuition from $\mathbb{R}^1 \to \mathbb{R}^1$ misleading.

Classic Non-Example: $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$(x,y) \mapsto \begin{bmatrix} e^x \cos y \\ e^x \sin y \end{bmatrix}$$

$\det(Df(x,y)) = e^{2x} \neq 0 \quad \forall (x,y) \in \mathbb{R}^2$. But f is not one-one since $\cos y, \sin y$ periodic of period 2π.

③ Corollary: Now compute derivative of f^{-1} using chain rule.

(now that we know f^{-1} defined and differentiable, under these assumptions)

$$\left[Df^{-1}(\underline{y})\right] = \left[Df(f^{-1}(\underline{y}))\right]^{-1}$$

since $f \circ f^{-1}(y) = y$.

Classic example: $e^{\ln x} = x$ where we think of $\ln x$ as inverse of e^x.

chain rule on left: $\frac{d}{dx}\left(e^{\ln x}\right) = \underbrace{e^{\ln x}}_{x} \cdot \frac{d}{dx} \ln x$

while $\frac{d}{dx}(x) = 1$, so $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

So far, have non-linear analogue of Theorem in Section 2.2. If A reduces to I_n in echelon form, then system $A\underline{x} = \underline{b}$ has unique solution for every $\underline{b}$. "Locally, f is behaving like linear function"

Now finish with ~~gen~~ non-linear version of general theorem on sol'ns to

$A\underline{x} = \underline{b}$. $\underline{b} \in \mathbb{R}^n$, $\tilde{A}$ has n pivot columns, m non-pivot columns.

Then can freely choose m non-pivot variables,

and these determine unique choice of n pivot variables giving a sol'n.

(Thm. 2.2.1, part 2b.)

Recall there's ambiguity in these terms. Pivotal / non-pivotal variables just depend on arbitrary ordering. (we list x_1 before x_2, etc.)

But number of them in each is intrinsic.

Generalization to non-linear equations is called "Implicit Function Theorem" – met this before, e.g.

unit circle : $x^2 + y^2 = 1$ Rewrite as : $x^2 + y^2 - 1 = 0$.

View x as independent variable, write $y = \sqrt{1 - x^2}$. Defines a function

on some nbhd. of (x_0, y_0) provided $(x_0, y_0) \neq (-1, 0), (1, 0)$.

Implicit Function Thm : $f : U \subseteq \mathbb{R}^{n+m} \to \mathbb{R}^n$, continuously diff.

s.t. $f(\underline{c}) = \underline{0}$, and $Df(\underline{c}) : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is onto.

(hence with n pivot vars, m non-pivot vars)

Then $\exists$ nbhd. of $\underline{c}$

in which $f(\underline{c}) = 0$ defines n dependent vars in terms of m

independent vars. via function g.

In our example $f(x,y) = x^2 + y^2 - 1 \;:\; \mathbb{R}^2 \longrightarrow \mathbb{R}$

$Df = 1 \times 2$ matrix $[2a, 2b]$ @ $(a,b) = \underline{c}$, on unit circle.

$[2a, 2b] \cdot \begin{bmatrix} x \\ y \end{bmatrix} \longmapsto 2ax + 2by$, so we can get any point in $\mathbb{R}$ by choice of (x,y) (remember, (a,b) fixed) provided $(a,b) \neq (0,0)$ (and its not because $(0,0)$ not satisfying $f(0,0) = 0$.)

If we want x indep. var, y dependent var. then need $b \neq 0$. (and then we can solve for y ~~as~~ a function of x)

Better: view y as a function of x.

There is a quantitative version of the implicit function theorem with Df Lipschitz satisfying precise inequality for Kantorovich's thm.

Remark: If we write our sol'n $\underline{c} \in \mathbb{R}^{n+m}$ to $F(\underline{c})$ with $\underline{c} = \begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix} \begin{matrix} \}\, n \text{ pivots} \\ \}\, m \text{ non-pivot vars.} \end{matrix}$

then the implicit function $g : \mathbb{R}^m \to \mathbb{R}^n$ is defined on a neighborhood of $\underline{b} \in \mathbb{R}^m$ with $g(\underline{b}) = \underline{a}$ and $F\begin{pmatrix} g(y) \\ y \end{pmatrix} = \underline{0}$ $\forall\, y \in B_R(\underline{b})$

magic R from quant. version.

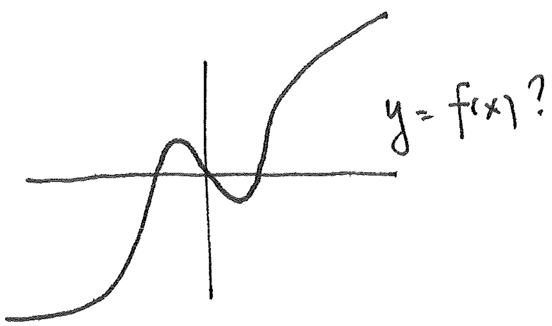
Corollary: By chain rule,

$$[Dg(b)] = - \left[D_1 F(\underline{c}), \dots, D_n F(\underline{c}) \right]^{-1} \left[D_{n+1} F(\underline{c}), \dots, D_{n+m} F(\underline{c}) \right]$$

Example from Krantz - Parks on Implicit function Theorem:

$$y^5 + 16y - 32x^3 + 32x = 0.$$

check from graph that it appears we can define y as a function of x.
(but not x as a function of y).



pf: Derivative in y for any fixed x is

$$5y^4 + 16 > 0.$$

so monotonically increasing

y^5 dominates so as $y \to \pm\infty$, LHS $\to \pm\infty$.

Apply intermediate value thm:

$\exists!\ y$ s.t. LHS $= 0$.

idea of implicit <u>function theorem</u> :

Suppose given $F : \mathcal{U} \subseteq \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ with $F(\underline{c}) = 0$.

and $DF(\underline{c})$ onto. Split up domain $\mathbb{R}^{n+m}$ into $\underline{x} \in \mathbb{R}^m$ (non-pivot)
$\underline{y} \in \mathbb{R}^n$ (pivot)

Consider new function f made from F: $f : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$

$$f\begin{pmatrix} \underline{x} \\ \underline{y} \end{pmatrix} = \begin{bmatrix} \underline{x} \\ F\begin{pmatrix} \underline{x} \\ \underline{y} \end{pmatrix} \end{bmatrix} \begin{matrix} \}\ \mathbb{R}^m \\ \}\ \mathbb{R}^n \end{matrix}$$

Use inverse function theorem at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \underline{c}$ with $F(\underline{c}) = 0$.

so construct inverse from nbhd. of $\begin{bmatrix} x_0 \\ 0 \end{bmatrix}$. Call it

$$g : \begin{bmatrix} \underline{x} \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} \underline{x} \\ \phi(\underline{x}) \end{bmatrix}$$

Where ϕ is defined according to this inverse g.

This is desired implicit function.