On Wednesday, learned that solutions to $A\underline{x} = \underline{b}$ (and # of solns) are determined by pivot columns in $[\tilde{A} \mid \tilde{b}]$, echelon form of $[A \mid b]$.

- if $\tilde{b}$ has a pivot, no solns
- otherwise, solns are in one-one correspondence with points in $\mathbb{R}^k$, $k$ = # of non-pivot columns.

Corollary: ~~If~~ $A$ reduces to id. in echelon form if and only if, for every $\underline{b}$, $A\underline{x} = \underline{b}$ has unique soln.

Pf: if $[A \mid \underline{b}] \underset{\text{row reduce}}{\rightsquigarrow} [I_n \mid \tilde{\underline{b}}]$ and unique soln is $\underline{x} = \tilde{\underline{b}}$.

A reduces to identity then

(solns preserved by row reduction)

If $A$ doesn't reduce to identity, H-H say that if $\tilde{A} \neq Id$ then either $\infty$-ly many solutions or no solutions. This is false, as stated, for a particular choice of $\underline{b}$.

ONE MORE CASE TO RULE OUT: We can have
$$\left[\begin{array}{ccc|c} 1 & & & \tilde{b}_1 \\ & \ddots & & \vdots \\ & & 1 & \tilde{b}_n \\ \hline 0 & \cdots & 0 & 0 \\ & \vdots & & \end{array}\right].$$

So just have to show that ~~...~~ if $\tilde{A} = \left[\begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \\ \hline 0 & \cdots & 0 \\ & \vdots & \\ 0 & \cdots & 0 \end{array}\right]$ then $\exists$ $b$ s.t. $[\tilde{A} \mid \tilde{b}]$ has no solns.

Row reduce for arbitrary $\underline{b}$: look at $\tilde{b}_{n+1}$ = linear comb. of $b$'s. say: $c_1 b_1 + \cdots + c_m b_m$. Pick $b_i$'s s.t. this equation is not 0. This is desired $\underline{b}$ with no soln. ✓

We're used to thinking about matrices acting on left of vectors:

$$A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{x_1 A(\vec{e}_1) + \cdots + x_n A(\vec{e}_n)}$$

Columns of $A$

But if we multiply by row vector on left:

$$[x_1 \cdots x_n] \cdot A = x_1 \cdot (\text{row } 1 \text{ of } A) + \cdots + x_n (\text{row } n \text{ of } A)$$

For example

$$[5 \ 1 \ 0 \cdots 0] \cdot A = 5 \cdot (\text{row } 1 \text{ of } A) + (\text{row } 2 \text{ of } A)$$

So

$$\begin{array}{cc} & m \\ m & \begin{bmatrix} 1 & & & & \\ 5 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \end{array} \cdot A$$

executes the row operation replacing row 2 of $A$ by $5 \cdot R1 + R2$.

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Of course scalar ~~mat~~ multiplication is easy to achieve by matrix multiplication.

Just use diagonal matrix:

$$\begin{array}{cc} & \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & c & & \\ & & & & 1 & \\ & & & & & \ddots \ \ 1 \end{pmatrix} \end{array}$$

$j^{th}$ row

to multiply row $j$ by $c$.

To swap rows, move 1's: e.g.

$$\begin{pmatrix} & 1 & & & \\ 1 & & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

swaps first two rows

Call these three types "elementary matrices". Compose them together to get a matrix $E$ that, when we multiply on the left by it,

then $E \cdot A$ is in echelon form.

Example from Wednesday: Matrix $A = \begin{bmatrix} 2 & 4 & 10 \\ 4 & 8 & 7 \end{bmatrix}$

row reduced to $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

by: $\rightarrow \begin{matrix} \frac{1}{2} R1 \\ R2 \end{matrix} \rightarrow \begin{matrix} R1 \\ R2 - 4R1 \end{matrix} \rightarrow \begin{matrix} R1 \\ -\frac{1}{13} R2 \end{matrix} \rightarrow \begin{matrix} R1 - 5R2 \\ R2 \end{matrix}$

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1/13 \end{bmatrix} \quad \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$$

Easy mnemonic for elementary matrices: Do the described op. to identity matrix.

$$E_4 \, E_3 \, E_2 \, E_1 \quad A = \begin{bmatrix} 2 & 4 & 10 \\ 4 & 8 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/13 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\underbrace{\begin{bmatrix} 1 & 5/13 \\ 0 & -1/13 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ -2 & 1 \end{bmatrix}}$$

check this

$$\begin{bmatrix} -7/26 & 5/13 \\ 2/13 & -1/13 \end{bmatrix} \begin{bmatrix} 2 & 4 & 10 \\ 4 & 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

More of a useful proof tool than a method...