Exploring consequences of definition of derivative: $Df(\underline{a})$: linear trans

such that $$\lim_{\underline{h}\to\underline{0}} \frac{f(\underline{a}+\underline{h}) - f(a) - Df(a)(\underline{h})}{|\underline{h}|} = \underline{0}.$$

Think of numerator as describing whether $f(a) + Df(a)\cdot h$ is a good linear approximation to $f(\underline{a}+\underline{h})$.

If one-variable f with power series rep'n, then we'd subtract constant, linear terms, leaving quadratic and higher, so indeed limit is $\underline{0}$.

Prove that differentiable $\Rightarrow$ continuous.

Directional derivatives

Properties of derivatives

(1) Derivatives of constant function $=$ 0 matrix

(2) Derivative of linear function is function (i.e. matrix of its linear transformation)

(3) Derivative of f exists at $\underline{a}$

$\Longleftrightarrow$ Derivative of $f^{(i)}$ $i=1,\dots,m$ exist at $\underline{a}$

then $$Df(\underline{a})\underline{v} = \begin{bmatrix} Df^{(1)}(\underline{a})\,\underline{v} \\ \vdots \\ Df^{(m)}(\underline{a})\,\underline{v} \end{bmatrix}$$

(4) sums, product of $\mathbb{R}$-valued and $\mathbb{R}^m$ valued, dot products

(5) all polynomials, rational functions w/ non-vanishing denom are differentiable

Additional remarks:

① We obtain lots of properties of derivatives from properties of limits

e.g. sums, products, composition,

f differentiable $\iff$ $f^{(i)}$ differentiable for $i=1,\dots,m$.

② Can compute directional derivatives – viewed as generalization of partial derivatives, now not along coordinate axis but along a vector.

Define $D_{\underline{v}} f(\underline{a}) = \lim_{t \to 0} \dfrac{f(\underline{a} + t\underline{v}) - f(\underline{a})}{t}$ (here t is a scalar in $\mathbb{R}$)

Proposition: If f is differentiable at $\underline{a}$, then

$$D_{\underline{v}} f(\underline{a}) = \underbrace{Df(a)}\cdot \underline{v}$$

Jacobian matrix mult. vector $\underline{v}$

pf: Check for $t \to 0^+$. Let you check $t \to 0^-$. (just as easy)

then $\lim_{t \to 0^+} \dfrac{f(\underline{a} + t\underline{v}) - f(\underline{a}) - Df(\underline{a})\cdot(t\underline{v})}{t} = \underline{0}$

since f diff. at $\underline{a}$. But $Df(a)$ is linear so

$Df(\underline{a})(t\underline{v}) = t \cdot Df(\underline{a})(\underline{v})$ $\Rightarrow$

$D_v f(\underline{a}) = \lim_{t \to 0^+} \dfrac{f(\underline{a} + t\underline{v}) - f(\underline{a})}{t} = Df(\underline{a})\, v.$ ✓

proof that derivative of linear map is itself:

cute idea: use uniqueness of $Df(\underline{a})$.

Indeed, f is linear so $$\lim_{\underline{h} \to \underline{0}} \frac{f(\underline{a}+\underline{h}) - f(\underline{a}) - f(\underline{h})}{|\underline{h}|} = \underline{0}$$

because numerator is identically 0. //

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other proofs of derivative laws amount to clever manipulation of difference quotient (e.g. add + subtract the same term to allow regrouping + factoring)

and resemble their one-variable counterparts.

chain rule : $g: \mathcal{U} \to V \quad \mathcal{U} \subseteq \mathbb{R}^n,\ V \subseteq \mathbb{R}^m$

$f: V \longrightarrow \mathbb{R}^p$ so $f \circ g$ defined.

Suppose g diff. at $\underline{a}$, f diff. at $g(\underline{a})$ then

$f \circ g$ diff. at $\underline{a}$ write derivative $D(f\circ g)(\underline{a}) =$

$Df(g(\underline{a})) \circ Dg(\underline{a})$

composition of linear transformations (matrix mult.)

pf. of chain rule : Kind of messy...

example : $f: \mathbb{R}^2 \to \mathbb{R}^2 \qquad g: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x^2 - y^2 \\ 2xy \end{bmatrix} \qquad \begin{bmatrix} u \\ v \end{bmatrix} \to \begin{bmatrix} u\cos v \\ u \sin v \end{bmatrix}$$

$$D(f\circ g)\begin{bmatrix} u \\ v \end{bmatrix} \underset{\text{Chain Rule}}{=} Df\left(g\begin{bmatrix} u \\ v \end{bmatrix}\right) \cdot Dg\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)$$

$$g\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u\cos v \\ u \sin v \end{bmatrix} \quad \text{and} \quad Df = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$$\text{so} \quad Df\left(g\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 2u\cos v & -2u \sin v \\ 2u \sin v & 2u\cos v \end{bmatrix}$$

$$Dg = \begin{bmatrix} \cos v & -u\sin v \\ \sin v & u \cos v \end{bmatrix}.$$

Now do matrix mult.

(can also just write composition explicitly and take Jacobian.

$$f\circ g \begin{bmatrix} u \\ v \end{bmatrix} = f\begin{bmatrix} u\cos v \\ u \sin v \end{bmatrix} = \begin{bmatrix} u^2 \cos 2v \\ u^2 \sin 2v \end{bmatrix}$$

using trig ident.

Check they are the same.