Peer Assisted Learning (PAL) Probability Unit Techniques

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1 Preface

Often times the Probability Unit is a difficult subject for the students enrolled in College Algebra and Probability. This handout is meant as a tool for facilitators to use to better prepare for sessions and be able to better redirect questions to help students help themselves. In the following sections, you will find tips and tricks to give your students to help them learn the material for themselves. Additionally, there are exercises that you can have the students do which are an extension of problems listed in the text. Remember that mathematics is first and foremost about *developing and explaining logic*; not number crunching. As you read this and develop your session plans, keep this in mind. Enjoy!

2 The Fundamental Principle of Counting

2.1 Key Terms and Concepts

- **The Fundamental Principle of Counting** p. 631/632 - (also known as *the multiplication rule*) If some overall process can be broken down into stages or parts, it is possible to multiply the number of outcomes at each stage together to obtain the total number of outcomes for the overall process.
- **Tree Diagram** p. 631 - a visualization method which shows all the outcomes for each stage and allows one to count the total number of outcomes. (For an example, see p. 631 of the text.) Only particularly useful if there are a small number of outcomes for each stage of the overall process.

2.2 The Process

There are two general types of problems in this section: subprocesses and ordering. For the subprocesses problems, there is a general approach. First consider these questions:

1. What is the *goal* of the problem, that is what the problem is asking you to find? (Equivalently, what is the overall process?)
2. What are the subprocesses taking place? How many are there?
3. How many choices are there for each subprocess?

Once we can answer these questions, we can apply the Fundamental Principle of Counting, multiplying all of the choices for the subprocesses together to obtain the answer.

In ordering problems you are trying to place a certain collection of objects into some order following some sort of guidelines. A couple of techniques are to create *superobjects* or to identify *options and trials*. In *superobject* problems, we can identify a certain procedure:

1. If two objects need to be placed together, put them together in a new object O. (If there are more than two objects, do the same procedure.)

2. As our task is to now place all the remaining objects and O together, we count up the number of ways we can place the objects.

3. After counting the number of ways we can place the objects, we need to multiply by the number of ways which the objects we put into O could be placed. (i.e. if A and B were placed into O then we need to multiply by 2 as we could have AB or BA in object O.) Once this is done, you will have arrived at your answer.

Notice that when doing this, you must reduce the number of slots – see below – by one less than the total number of objects combined.

In *options and trials*, your primary goal is to determine what to classify as "options" and what to classify as "trials", for after that is completed, you can say that your answer will be

$$Options^{Trials}$$

by the Fundamental Principle of Counting. "Trial" objects are generally objects that you can place into/onto another object and "Option" objects are generally all possible outcomes – the number of objects that the trial objects could end up in or at. (Simply thinking of something in terms of options and trials works too!)

2.3 PAL Applications and Techniques

For subprocess problems, a good way to remember how to use the Fundamental Principle of Counting in these scenarios is through the use of the acronym *G.P.S.*:

- **G**oal - determine the goal of the problem
- **P**rocess - identify the overall process
- **S**ubprocesses - determine the subprocesses and number of choices for each subprocess

Pictures also help as well. In addition to the tree diagram mentioned in Section 10.1 of the text, the idea of **slots** comes in particularly useful in probability if you must *order* some group of objects. To use slots, simply draw dashes – enough dashes for the number of objects or stages that you have. After doing this, place the number of choices for object/stage one in the first slot, the number of choices for the second object/stage in the second slot, and so on until all the slots are filled. Then simply multiply between slots and you will have obtained your answer. We will see these in some examples.

2.4 Examples

Example 1 A certain family decides to go out to dinner at a 5 course restaurant. After getting the menu, the family decides it would be interesting to figure out the total number of possible meals they could have. They count that there are 6 different appetizers, 3 different salads (excluding dressing choices), 2 different soups, 15 different entrees, and 5 different dessert choices. How many different meals can be made with these choices?

Solution: This is a subprocess type problem, so we use that method to find the answer. Our ***goal*** is to count the total number of meals. That being said, we can say the ***overall process*** is the meal itself. Each meal has 5 stages, which are our ***subprocesses*** - the appetizer, salad, soup, entree and dessert. So let's think of the different stages as *slots*. As we said the stages are like slots, we can say that

$$No.Meals = No.Appetizers * No.Salads * No.Soups * No.Entrees * No.Desserts$$

by the Fundamental Principle of Counting, where the first slot is for the number of different appetizers, the second slot is for the number of different salads, and so on through dessert. So now all we have to do is to place the appropriate numbers in the slot, remembering to multiply between the stages and we will have reached our goal!

$$6 * 3 * 2 * 15 * 5 = 2,700$$

Example 2 Minnesota License Plates contain three letters followed by a three digit number. How many license plates are possible that exclude the letters Q, X, and Z where letters cannot be repeated, and also contain only even numbers that are greater than 299 (numbers can be repeated, zero is neither even nor odd)?

Solution: Try using slots to solve this problem. We know that there are six total slots – three for letters followed by three for numbers. Then by the Fundamental Principle of Counting we have:

$$L_1 * L_2 * L_3 * N_1 * N_2 * N_3 = Total$$

Where "L" denotes a letter and "N" denotes a number. We know there are 26 letters, but we also are given that there cannot be Q, X, or Z in the license plate, leaving 23 options for the first letter L_1, 22 for the second letter L_2, and 21 for the third letter L_3. Now we work on the numbers: the numbers 300-999 are available, but we can only use those that are even. If we analyze the numbers by digits, the first digit N_1 can be 3-9 (7 options), the second digit N_2 can be 0-9 (10 options) and the third digit N_3 can be 2, 4, 6, or 8 (4 options). So our previous equation under the constrictions gives

$$23 * 22 * 21 * 7 * 10 * 4 = 2,975,280$$

Example 3 Akram, Beatrice, Cindy, Dave, Edgar, Francis, George, and Hector go to a movie theater and want to sit down in a row of eight seats such that Dave and Edgar are next to each other. How many ways can this be done? How many ways could Dave, Edgar, and Cindy sit together (and the others occupying the other seats)?

Solution: This is a superobject type problem. Let's tackle the case where Dave and Edgar sit together. According to what was described previously, we want to make them sit together, so we make them into one object, so instead of eight places to sit, we have seven as Dave and Edgar condensed into one object/person. Next, we can make slots like the following:

$$Total = S_1 * S_2 * S_3 * S_4 * S_5 * S_6 * S_7$$

where S denotes the seat position. Now we just count options for each position. S_1 will have 7 options as no one has sat yet, S_2 will have 6 options as one person has sat down, and so on until there is only 1 option for S_7. But wait...we forgot to multiply by the number of ways in which Dave and Edgar could sit! Fortunately there are only two ways: Dave then Edgar, or Edgar then Dave. So our previous equation becomes:

$$Total = (7 * 6 * 5 * 4 * 3 * 2 * 1) * 2 = 10,080$$

Now we try to solve the second part of the question. First we place Dave, Edgar, and Cindy into one object, call it O. Now we have to place Akram, Beatrice, Francis, George, Hector, and O - so we have 6 total slots:

$$Total = S_1 * S_2 * S_3 * S_4 * S_5 * S_6 = 6 * 5 * 4 * 3 * 2 * 1 = 720$$

But again we need to multiply this by the number of ways in which Dave (D), Edgar (E), and Cindy (C) could sit together. There are six ways: DEC, DCE, EDC, ECD, CDE, CED. So our answer is:

$$Total = 720 * 6 = 4,320$$

Example 4 Suppose we flip a fair coin (contains only heads and tails) a total of 5 times. How many different outcomes could we obtain?

Solution: Here we have exactly two options that can occur at the end of a flip (heads or tails) and we repeat the process of flipping 5 times (the number of trials). This is an *Options and Trials* type problem from the Fundamental Principle of Counting. This process tells us that there will be $2^5 = 32$ different outcomes. (Alternative solution that is omitted would be to utilize a tree diagram that lists all the different options.)

2.5 Exercises

1. Ramilda wants to grow a vegetable garden, but the garden is only large enough for one type of plant to grow in it. Before she can plant her garden, she needs to decide on what type of vegetable to grow, the type of fertilizer to use, and the type of cage she should use to protect the plant from pests. Her choices of vegetables are tomato, cucumber, onion, carrot, radish, and string pea. As for fertilizer, she can either use Miracle-Gro or the generic brand. The cages she can choose from are square chicken wire, circular chicken wire, square metal, and circular metal. How many possible gardens could Ramilda have? *Answer: 48; Textbook problems: 1, 2, 7, 8, 9; Hint: Subprocess*

2. Juan has a lot of homework to do tonight. He has to complete homework for 4 different classes: Calculus, Economics, Psychology, and Chemistry. Furthermore, each class has a different number of problems that he must complete. In total he has 8 Calculus exercises, 5 Economics short answer questions, 4 Psychology short answer questions, and 10 Chemistry problems. In how many different ways could he complete the problems? *Answer: 1600; Textbook problems: 1, 2, 7, 8, 9; Hint: Subprocess*

3. A chemist wants to keep track of the successes and failures in his upcoming experiment. If he repeats the experiment 10 times, denoting the outcome of the experiment as either a success or a failure, how many possible success-failure strings of results could he get? *Answer:* $2^{10} = 1024$*; Textbook problems: 10, 20, 21, 24, 25; Hint: Options and Trials*

4. Analise, Ben, Cathy, Danielle, Eric, Frank, and George go to see a movie. In the theater, Analise and Frank want to sit together. How many possible ways can the 7 people sit in seven seats under this restriction? *Answer: 1440; Textbook Problems: 15-20; Hint: Superobject, Example 3* Now Cathy and Danielle also want to sit together. How many possible ways can the group sit together? *Answer: 480*

5. University of Minnesota Student IDs are 7 digits long. How many IDs are possible with the numbers 0-9 if no number can be repeated? *Answer: 10*9*8*7*6*5*4=604,800; Textbook problems: 11, 29, 30; Hint: Slots, Example 2* Suppose now they only want to use the numbers 0, 1, 2, 3, 4, 8, and 9. How many IDs are possible if the office is only concerned with ID where the first digit is greater than 5, again no number can be repeated? *Answer: 2*6*5*4*3*2*1=1440*

3 Permutations and Combinations

3.1 Key Terms and Concepts

- **Factorial Notation** p. 637 - a value followed by *!* denotes the product of that number and all preceding values, ending at 1. i.e. $5! = 5*4*3*2*1$

- **Permutations** p. 638 - arrangements in which order matters. $P(n,r)$ denotes the number of ordered arrangements of n objects into r distinct (different) positions.

- $P(n,r) = n(n-1)(n-2)...(n-(r-1))$ p. 638

- **Multinomial Theorem** p. 640 - If an object of length n can be decomposed into elements such that there are r_1 things that are of the same kind, r_2 things that are of the same kind and so on until the r_k term, then the total number of distinguishable permutations is $\frac{n!}{(r_1)!(r_2)!...(r_k)!}$

- **Combinations** p. 641 - arrangements in which order does not matter. $C(n,r)$ denotes the number of ways of selecting r elements from n elements.

- $C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$ p. 641

- **Order matters versus Order does NOT matter** – to be discussed further

3.2 The Process

In these problems there are essentially two steps:

- *Identify* the total number of objects available and the number of objects you will actually use.
- *Decide* whether or not order matters. If we can reach a conclusion on whether or not order matters, we have decided whether to use permutations or combinations and can complete the process by the appropriate counting method.

To decide whether or not order matters, the main question to ask is, after we select the objects that we want to place into the available positions to count, do we have **different labels** for each position? That is, if we were to place one object in a certain position, would it be different than if we were to place it in another place? If yes, then we use permutations to count as the order matters; if not, use combinations. What if some of the objects are indistinguishable? Then we can use combinations as order doesn't matter, however a more convenient trick is to use the *Multinomial Theorem*, IF we have some indistinguishable items AND distinguishable items we need to place.

3.3 PAL Applications and Techniques

As a PAL Facilitator, students may often ask whether or not order matters. There are some questions and ideas to give the students:

- As was noted in the previous section, if we change the order of the result, will it make a difference? (If it does, then use permutation to calculate the number of ways; if not, then use combination)
- Do the objects become labeled differently in the end? (i.e. First, Second, Third,... or President, Vice President, Secretary, ... etc.) If so, then order of the objects *does* matter.
- Make slots as done for the Fundamental Principle of Counting and place the objects that you pick into the slots. Before counting the number of ways via the Fundamental Principle of Counting, do the slots have labels? If so, that means order matters.
- Make a tree diagram like on p. 631 or list out all possibilities if the number is small.

3.4 Examples

Example 1 A marathon has 50 participants. In how many ways can the first, second, and third, fourth and fifth place prizes be given out?

Solution Here we are concerned with the first five places of the winners so let's make slots as we did in the Fundamental Principle of Counting section:

$$Total = First * Second * Third * Fourth * Fifth$$

(We have a slot for each place.) Next, as the participants are different than each other, we can name them somehow (this part is not vital to the overall process), but let's say that Alice (A), Barbara (B), Chuck (C), Denise (D), and Eugene (E) are the first five to finish the marathon. As we have picked our objects to place, we become stuck – do we count the total via permutations or combinations? Well, let's say we pick to count them with combinations – so order should not matter. Then if we have the order A,B,C,D,E occupying first through fifth places respectively, then that should be the same as E,D,B,C,A. Let's look at Alice (A). She would be quite upset if she received the fifth place award if she actually came in first, so order *does* matter here! That means we count using permutations and we have

$$P(50,5) = 50*49*48*47*46 = 254,251,200$$

Example 2 Suppose a certain club has 15 members, 6 of which are male and 9 of which are female. The club wants to undertake a project which requires 6 people and they want the committee to have an equal number of male and female members. In how many ways could the club select members for the committee?

Solution We know that we have 6 positions available on the committee, so we need to select our members to fill the committee. Then we have committee (C) composed of six members (M) and so we can make the following diagram

$$C = M_1 * M_2 * M_3 * M_4 * M_5 * M_6$$

by the Fundamental Principle of Counting. Before we fill the committee, we must decide on how we want to count the number of committees we could make: do we use permutations or combinations? We know all the members picked will be a part of the committee, and to our knowledge, all of the members will have the same duty - so it really would not matter who goes where in the committee. From this, we determine that order of the members, once selected, does not matter and so we use combinations. As the members are not ordered, three of the members must be male and three must be female, so we can use combinations to select the members so we can say the above formula changes to:

$$C = N_M * N_F$$

where $N_M = C(6,3)$ is the number of ways to select three males and $N_F = C(9,3)$ is the number of ways to select three females to serve on the committee. Hence we find the total number of ways to be

$$C = C(6,3) * C(9,3) = 20 * 84 = 1,680$$

3.5 Exercises

1. Publisher's Clearinghouse has checks for one million dollars, one-half million dollars, and one-quarter million dollars they are delivering to three different houses. Unfortunately, all the computers crashed and no one remembers which check goes to which house! In how many ways could they deliver the checks? *Answer* $P(3,3) = 3! = 6$

2. You and 9 friends (10 total people) decide to eat at a Chinese restaurant. As you enter and ask for a table, the host(ess) tells you that the largest table they have seats seven people. How many different ways could the group of 10 occupy the table that seats seven people? (The other three would sit at a nearby table.) *Answer:* $C(10,7) = C(10,3) = 120$*; Text Problems 21-26, 35-42*

3. How many different "words" can be made with the word FACILITATING? *Answer:* $\frac{12!}{(2!)(3!)(2!)} = 19,958,400$*; Hint: Multinomial Theorem; Text Problems 27-34*

4. The National Football League (NFL) consists of 32 teams. How many ways can the teams occupy the top twelve spots to be able to play in the playoffs? *Answer* $P(32,12) = 1.08x10^{17}$ Of those teams who make it into the playoffs, in how many ways can the top two teams be decided to play in the Superbowl? *Answer:* $P(12,2) = 132$

5. A certain small high school is celebrating Homecoming. As a part of the festivities, each grade must divide their students among window decorating and float building. However, all candidates for Homecoming King and Queen cannot take part in these activities, but have other things to do during the time. The senior class has 60 students and 8 of those are candidates for King or Queen. If there can be a total of 25 students doing window decorating (the rest going to float building), how many ways can the students be divided up? *Answer:* $C(52,25) = C(52,27) = 4.78x10^{14}$ Meanwhile the King and Queen Candidates are supposed to be designing signs and other materials to boost school spirit, but instead they are figuring out all the possible pairings of Kings and Queens. If there are four candidates for each position, how many different

pairings are there? *Answer: 16* Now the school also allows there to be a Prince and Princess. How many King-Queen-Prince-Princess combinations are there? *Answer: 144*

4 Probability

4.1 Key Terms and Concepts

- **Event Space/Event, $n(E)$ or E,** p. 646 - outcomes of a process under given restrictions.
- **Sample Space/Total Sample, $n(S)$ or S,** p. 646 - total number of outcomes of a process with AND without the constraints specified in the problem.
- **Probability** p. 646 - The probability that event E will occur is defined as

$$P(E) = \frac{n(E)}{n(S)}$$

or simplifying the notation, we can say,

$$P = \frac{E}{S}$$

4.2 The Process

By now, we have the necessary counting tools to determine both E and S:

1. For S, determine what process is being done.
2. Decide whether order matters or not in that process; if it does, then count using permutations, if not, use combinations.
3. After counting S, we turn to count E. There are no hidden tricks in counting E; we just need to be sure to count correctly, which is often the most difficult part. Decide how to count E based on known methods: apply the Fundamental Principle of Counting, permutations, or combinations.
4. Now knowing S and E, we are able to calculate P.

At this point our English skills become very important as certain words in English have key mathematical interpretions. In the following section, tactics to count the event are explored.

4.3 PAL Applications and Techniques

As previously mentioned, calculating E is difficult for students. Here are some tips and questions to ask them as a facilitator (and yourself as well, as you prep for the session):

- What do *at least* and *at most* signify in a problem? **“At least” means we take the case that it says as well as any cases that are greater than or after it. “At most” means we take the case in question and all preceding cases as well into account.**
- Does the word *or* mean add or multiply in mathematics? **Add. Remember that the Fundamental Principle of Counting results in a multiplication as the whole process can be done in stages. An “or” means that the entire process could be done “this way”, “that way”, etc. (This idea is *mutual exclusivity*, something we will look at in the next section.) As a result we must add the different ways the event could take place.**
- Does the word *and* mean add or multiply in mathematics? **Multiply. “And” is adding another step or condition to the problem, which is like another stage we have to go through in the problem. The Fundamental Principle of Counting tells us that stage processes result in a multiplication.**

- Is there another way to count the event? If we have two different things that make up the total sample, when we count one thing, we count the other (i.e. if we want to count coin flips that end in tails, we automatically count heads as well). This is known as the *Complement Rule*. If there is another way, do our two answers agree? **The answers should agree.**
- If there are conditions given for the event, let them occur *first* in any diagrams you create and after this count the number of ways of filling the remaining spots.
- Is there more than one way to create a diagram of the event we are to count? If so, what do we do with the results – add or multiply them? **We add them. They are different cases of the same thing, so we treat them like an "or" and add the results.**

For those who struggle calculating S, there are two main questions that often help:

- Is the process being done in stages? If so, how do we apply the Fundamental Principle of Counting?
- Does the order in which the objects occur matter? Do we use permutations or combinations to count these objects?

4.4 Examples

Example 1 Suppose you toss a fair coin 7 times. What is the probability that there are exactly 5 heads? At most 3 tails?

Solution To begin, let's identify our S, our sample space or total sample. For each coin there are 2 options - heads and tails - and we have 7 tosses. This is a "Options and Trials" type problem from the Fundamental Principle of Counting, so $S = 2^7 = 128$. Now for each event - let's consider 5 heads first. Our first question here is does the order in which we flip the heads matter? No - we are just concerned with flipping 5 heads. As such we decide to count with combinations. We have seven places (diagram is omitted) and we want five of them to be heads, so $E = C(7,5) = 21$ so our answer for this first part is

$$P = \frac{E}{S} = \frac{21}{128}$$

But wait, can we check our answer another way? YES - we can count the number of *tails* instead! If we have five heads, that means we must have two tails, so $E = C(7,2) = 21$ (Note: notice something curious here?! Think about Pascal's Triangle!!) which gives us the same answer, so our methods check out!

Now we move to the second part. As mentioned before, if we have "at most", we must take the case in question and all preceding cases. Hence, here we must consider 0 tails, 1 tail, 2 tails, and 3 tails. We still have the same number of flips, and so our S remains the same. Now we move onto calculating E. We said in the previous part that order doesn't matter and the same is true here, so we use combinations to count E. For 0 tails we have $C(7,0) = 1$, 1 tail is $C(7,1) = 7$, 2 tails is $C(7,2) = 21$, and 3 tails is $C(7,3) = 35$. Now as these are different possibilities, we must add to obtain E as we could get 0 tails OR 1 tail OR 2 tails OR 3 tails. (Remember OR in mathematics means add.) So we find $E = 1+7+21+35 = 64$ and then the probability here becomes

$$P = \frac{E}{S} = \frac{64}{128} = \frac{1}{2}$$

The reader may confirm that the same answer is obtained by performing similar calculations for the number of heads in each scenario.

Example 2 Zach, Ye, Xenia, Will, Ulysses, Tricia, Samuel, Rosa, and Quentin want to volunteer at a local food shelter. The coordinator at the food shelter needs five people to form a committee that will hand out food to people who walk in. What is the probability that Will and Xenia will be on committee? What is the probability that either Tricia or Ye, but not both, will be on the committee?

Solution We begin by computing S as it will be the same for both cases. S will be the number of ways that we can select 5 people out of the 9, but we need to decide if order matters or not. As there are no roles - meaning no labels - within the group of five, order will not matter, so we compute S using combinations and we find $S = C(9,5) = 126$. Note: we could have equivalently computed the number of ways to select the 4 people that will not be on the committee and would have received the same number.

Next, we count the committees that Will and Xenia will be on and this will be our first E. Before we do any placing of members, will order matter here or not? By the argument previously mentioned, order will not matter as there are no labels or roles within the group. So, we first let Will and Xenia occupy two slots in the committee and this is $C(2,2) = 1$. Next we must fill the other three slots on the committee with 3 of the remaining 7 people and the number of ways of doing this is $C(7,3) = 35$. As this is done is stages, we apply the Fundamental Principle of Counting and see that $E = C(2,2) * C(7,3) = 1 * 35 = 35$. In doing so, the probability that Will and Xenia will be on the committee is

$$P = \frac{E}{S} = \frac{35}{126} = \frac{5}{18}$$

Now let's count the committees where Tricia or Ye, but not both, are on the committee. First we notice the *or* – we will have an addition! Order will not matter here either, so let Tricia be on the committee and we need to pick four other members, one of which is not Ye, to join Tricia. The total possible members to join Tricia is 7 and we are picking 4 of them to join her – this is $C(7,4) = 35$. Now we let Ye be on the committee and pick 4 members to join him, which is $C(7,4) = 35$ as well. As we have the *or*, we add the two cases and we find that $E = C(7,4) + C(7,4) = 35 + 35 = 70$. Thus, the probability that either Tricia or Ye, but not both, will be on the committee is

$$P = \frac{E}{S} = \frac{70}{126} = \frac{5}{9}$$

4.5 Exercises

1. Katrina, Langdon, Mary, Nancy, Otto, and Peter have entered a Science Fair competition individually, with no other competitors. If prizes are awarded for first, second, and third places only, what is the probability that Langdon takes first place, Nancy takes second place, and Katrina takes third place? *Answer:* $\frac{1}{P(6,3)} = \frac{1}{120}$

2. A lightbulb company produces 20 lightbulbs in 10 minutes and five of them are known to be defective. Suppose the company takes a random sample of of 5 lightbulbs after 10 minutes. What is the probability that the sample contains 3 defective lightbulbs? *Answer:* $\frac{C(5,3)*C(15,2)}{C(20,5)} = \frac{1050}{15504} = \frac{175}{2584}$; *Text Problems 39-41*

3. A pair of dodecahedral (12-sided) dice labeled with the numbers 1-12 are tossed. What is the probability that the sum of the top-most faces is 17? *Answer:* $\frac{8}{144} = \frac{1}{18}$; *Text Problems 13-22*

4. A fair coin is tossed 10 times. What is the probability that at least 7 heads are flipped?
   Answer: $\frac{C(10,7)+C(10,8)+C(10,9)+C(10,10)}{2^{10}} = \frac{11}{64}$; *Text Problems 1-12, 54-57*

5. A mailman has five houses left on his route and 9 letters to deliver yet, dividing the 9 letters among the houses according to address. However, upon further inspection, he realizes that some water spilled on the letters and smeared the addresses on them to make the addresses indecipherable. Also instead of taking them back to headquarters and returning them to the sender, he decides to deliver them to the remaining houses in a random order. What is the probability that he decides to deliver 2 cards to four houses and only 1 card to the fifth? *Answer:* $\frac{C(9,2)*C(7,2)*C(5,2)*C(3,2)*C(1,1)}{5^9} = \frac{4536}{390625}$; *Text Problem 48*

5 Properties of Probability and Expected Values

This section covers a great many important ideas briefly - Set notation, Union/Intersection of Sets, The Complement Rule (Complementary Events), Elementary Inclusion-Exclusion, Mutually Exclusive Events, and Expected Value. Although these ideas take some time to become familiar with, none of them are particularly challenging to apply. Often times here, the trick is just to sift through the information in a problem to figure out what it is asking for and what process you need to apply.

5.1 Key Terms and Concepts/The Process

Let's explore the logic behind each of the ideas mentioned in the previous subsection:

- **Set Notation and Union/Intersection** - Sometimes students are quite familiar with set notation and sometimes they are not. Recall that the symbol, $\cup$, means "*union*" which takes the entire part of the objects on either side of the symbol and adds them together. (Figure 10.3 p. 655 in the book is a good demonstration.) On the other hand, $\cap$, means "*intersection*" and takes only the part where the objects on the left and right of the symbol overlap. (Figure 10.2 p. 655 in the book shows this clearly.) Often the trick is to figure out when to use each one. This will be visited in the next section.

- **The Complement Rule** p. 653 - If some set is composed entirely of *two* options (Heads/Tails, True/False, Yes/No, etc.) we can use the *Complement Rule* to count both of the objects in the sets, given that we know one of the two options. If we let E denote the option that we know and E' denote its complement, or the other option in the set, then
$$P(E) + P(E') = 1$$
as the probabilities must sum to one and the two events, E and E' are the only two possible outcomes in the set. In problems, the trick to to figure out what to label as E and E'. Also note that it is possible to rearrange the above equation to different forms, solving for one probability or the other depending on what is most convenient.

- **Elementary Inclusion-Exclusion** p. 655 - Further classes in Probability will teach the logic behind the formula given in the book. If we have two sets, say A and B, and we want to find the probability of their union, then we can represent that by the sum of the probability of set A, $P(A)$, and the probability of set B, $P(B)$, BUT if the two sets overlap, we have counted the intersection twice, so we must subtract the probability of the intersection of A and B, $P(A \cap B)$. That gives us
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
In these problems, A and B are usually easy to distinguish as are their probabilities, but the trick lies in identifying the other two. Our English skills become important again as well as analyzing the problem – more to come in the next subsection.

- **Mutually Exclusive Events** p. 656 - Recall that sets A and B are mutually exclusive if $P(A \cap B) = 0$ OR, equivalently, $A \cap B = \emptyset$, that is they do not overlap or intersect at all. For those that are more graphically-inclined, this would be the scenario where if both sets are modeled by circles, you can draw either circle *without* touching the other circle. The concept has a clear application in Inclusion-Exclusion, which reduces the equation to
$$P(A \cup B) = P(A) + P(B)$$
Mutual Exclusivity has more applications in the next section on Conditional Probability and Independence.

- **Expected Value** p. 658 - Often we want to find out whether or not something is worth our chances partaking in or not, which is where expected value comes in. For any event, we can compute the product of the *each* outcome and its respective probability. Summing all the products yields the expected value:
$$E_v = x_1 p_1 + x_2 p_2 + x_3 p_3 + \ldots + x_k p_k$$

for k different outcomes. The event is *fair* if the expected value E_v is equal to the amount charged to partake in the event.

5.2 PAL Applications and Techniques

To solve these problems, we can often go back to the methods we used in previous sections. Beyond this and redirecting questions, here are some questions to consider:

- What does $\cup$ mean in English – "And" or "Or"? **Or. As the symbol means "union", it means the collection of the two sets, so the object in question could be in either set.**
- What does $\cap$ mean in English – "And" or "Or"? **And. Intersection means that the object must lie in the two sets, or, rephrasing, the object must lie in Set A AND Set B.**
- How many outcomes are possible? If there are just two, if we count one, do we indirectly count the other? If yes, what can we use? **The Complement Rule**
- What techniques did we use to solve previous problems about coin flips and dice?
- Can probabilities be negative? **No. Property 10.1 on page 652 of the textbook states that probabilities must be between 0 and 1 inclusive. Since probabilities cannot be negative, but you are in a situation that you are pretty sure includes negative values, what else could be negative or how else could you account for a negative value?**
- If you are having trouble identifying what process to use, write down what you think are the important parts in the problem, attempt to identify what the different parts mean mathematically/in the context of probability. Then in a moment, compare with a neighbor and then another group. [*Activity: Think-Pair-Share*] What process(es) would work the best with the information you are being given and could produce what the problem is asking?

5.3 Examples

Example 1 In an election at Facilitator University, 5,000 students participated to elect one of two candidates for Student Body President. The two candidates, who wish to now remain anonymous, are called A and B. Of the voters, 2,800 voted for A and 2,400 voted for B. However some students were tricky and clearly voted for both A and B – the number of which was 600. Find the probability that if a student was selected at random from the 5,000 that they would have voted for either A or B.

Solution First, organize the information that is given:

- number of voters $= 5,000$
- voters for A $= 2,800$, so probability of a voter voting for A $= \frac{2800}{5000} = \frac{14}{25} = 0.56$
- voters for B $= 2,400$, so probability of a voter voting for B $= \frac{2400}{5000} = \frac{12}{25} = 0.48$
- voters for A and B $= 600$, so probability of a voter voting for A and B $= \frac{600}{5000} = \frac{3}{25} = 0.12$

Now we decide how we can use the information to obtain our desired information. Remember, our goal is to find the **probability** that a student votes for A **or** B – or interpreting it mathematically, this is the same as $P(A \cup B)$. Then from the above information that we calculated, $P(A) = 0.56, P(B) = 0.48$, and $P(A \cap B) = 0.12$. Then we can use Elementary Inclusion-Exclusion as we discussed before to obtain our result!

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.56 + 0.48 - 0.12 = 0.92$$

Example 2 In a certain casino, there is a modified game of roulette. To play, a player is required to pay 10 dollars. On the wheel there are 20 slots. The number and values of the different slots on the wheel are as follows: 6 slots worth zero dollars, 5 slots worth 5 dollars, 4 slots worth 10 dollars, 3 slots worth 20 dollars, 2 slots worth 30 dollars. Is this a fair game?

Solution To determine whether a game is fair or not, we calculate the mathematical expectation, or expected value of the game; it will be fair if the expected value is equal to 10 dollars, otherwise it will be unfair. In the game we have five different outcomes: 0 dollars, 5 dollars, 10 dollars, 20 dollars, and 30 dollars - these will be our x_i values:

- x_1 will denote the value of '0' slots $= 0$
- x_2 will denote the value of '5' slots $= 5$
- x_3 will denote the value of '10' slots $= 10$
- x_4 will denote the value of '20' slots $= 20$
- x_5 will denote the value of '30' slots $= 30$

To find the expected value, we need the different outcomes *and* their probabilities. We know the number of each kind of slot as well as that there are 20 slots in total. This is enough to find the probabilities of each kind of slot:

- p_1 is the probability of landing on a 0 (or getting x_1) $= \frac{6}{20} = 0.30$
- p_2 is the probability of landing on a 5 (or getting x_2) $= \frac{5}{20} = 0.25$
- p_3 is the probability of landing on a 10 (or getting x_3) $= \frac{4}{20} = 0.20$
- p_4 is the probability of landing on a 20 (or getting x_4) $= \frac{3}{20} = 0.15$
- p_5 is the probability of landing on a 30 (or getting x_5) $= \frac{2}{20} = 0.10$

With this, we are able to calculate the expected value:

$$E_v = x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 + x_5p_5 = (0)(0.30) + (5)(0.25) + (10)(0.20) + (20)(0.15) + (30)(0.10) = 9.25$$

Since 9.25 is not equal to 10.00, which is the amount paid for the game, so we conclude that the game **is not fair/is unfair.**

5.4 Exercises

1. A standard die (cube, 6 sided, faces have values 1-6) is tossed. What is the probability that a 3 or 6 is not rolled? *Answer:* $\frac{2}{3}$*; Hint: Complement Rule; Text Problems 17-19*

2. A science class has 18 students working on an experiment, 13 students working on a worksheet. If there are 25 students enrolled in this particular science class, how many students must be both working on an experiment and working on a worksheet simultaneously? Hint: This is Inclusion-Exclusion without incorporating probabilities. It can be done exactly the same as with probabilities. Conversely, if you choose to incorporate probabilities, be sure to multiply your result by 25 to obtain the number of students you desire. *Answer: 6; Text Problems 37-40*

3. A teacher takes red, green, blue, yellow, purple, white, and black beads to class in a bag for arts and crafts. However on the way to school, the container that separates the beads opened and all the beads mixed together. If there are 50 red, 30 green, 25 blue, 30 yellow, 15 purple, 35 white, and 15 black, what is the probability that when the teacher reaches in the bag that a white or blue bead will be drawn? *Answer:* $\frac{35}{200} + \frac{25}{200} = 0.3$ *; Text Problems 32-36*

4. Suppose you roll a standard die three times. After the three rolls, what is the probability that the top most faces yield a sum greater than 5? Hint: Try using the Complement Rule. We have two probabilities in the entire sample space – the probability of a sum greater than 5 and the probability of a sum less than or equal to 5. *Answer:* $1 - \frac{1+3+6}{6^3} = \frac{103}{108}$ *; Text Problems 1-8*

5. A lottery game has several different cash prizes (in dollars) - 5, 10, 20, 50, 100, 1,000, and 5,000. If 15,000 people enter the lottery, each paying a dollar, and there are 20, 20, 10, 10, 10, 8, and 1 of each prize respectively, is the lottery fair? *Answer: Yes it is fair; Text Problems 48-55*

6 Conditional Probability and Independence

At this point, all the previous material comes into play, so not only are students working with the new material, but they are also reviewing material from the entire chapter. As some students may not be actively reviewing the material, this could be another obstacle that you face as a Facilitator.

6.1 Key Terms and Concepts

- **Conditional Probability,** $P(A|B)$**,** p. 663 - the probability that event A occurs given/provided that B has already occurred.
- $P(A|B) = \frac{P(A \cap B)}{P(B)}$ p. 664
- **Independent Events** p. 665 - when two events say, A and B, can occur without affecting the results of the other. (Note: if they are not independent, they are *dependent*)
- A and B are independent if and only if $P(A \cap B) = P(A)P(B)$ p. 666

6.2 The Process

As with the last section, the most important thing is to read the problem carefully and identify everything in the problem – for hints on this, revisit sections 4.3 and 5.2. In many problems we can follow these steps:

1. **Determine** what are we to compute or find. What is this in mathematical terms?
2. **Interpret** the information given in the problem. What is the “A” and “B”? What else is given to us and how do we represent that information mathematically?
3. **Check** the problem to make sure all the important information has been taken from it.
4. **Locate** the proper method or formula to use with the given information. If there is any information that is needed for the method, try using previously acquired techniques from the Probability Unit to gather that information.
5. **Grind** out the number(s) via the method you chose. Recheck your calculations to ensure that your answer is correct.

What if we do not know the value of $P(A \cap B)$? We do have enough tools to figure it out! Remember Elementary Inclusion-Exclusion – we are able to solve for $P(A \cap B)$. Also remember *mutual exclusivity* - for mutually exclusive events A and B, $P(A \cap B) = 0$. (Most of the problems here will not use this condition though as they want to develop skills related to computing $P(A \cap B)$.)

Common examples of independent events are coin flips and dice rolls – each time they are done, the outcomes do not depend on the previous outcome. On the other hand, dependent events are pulling marbles out of a bag or cards out of a deck *if* they are pulled *without replacing* the object pulled before it (more to come in the next section). Often independence problems concern themselves mutually exclusive events and utilize the Fundamental Principle of Counting as they may ask for a probability certain outcome, but *do not* specify order in which the outcome must occur. As such, it is often necessary to *think of all possible orderings* the specific outcome could take place. After determining the orderings, the Fundamental Principle of Counting must be applied to compute the probability and then as the events are mutually exclusive, meaning that they are completely different orderings, you must **sum** the probability of the possible orderings to obtain the probability of the total outcome.

The equations in the section are straightforward computationally, but the trick is in determining each component with the given information.

6.3 PAL Applications and Techniques

In this section, there are a couple of phrases that are vital to understanding - *with replacement* and *without replacement*.

- "With replacement" implies that the events are independent. If they are independent, does anything change between events? **No - one event does NOT affect the other(s).**
- "Without replacement" means that the events are dependent. If they are dependent, does anything change between events? **YES! If we repeat a process without replacement, the total number of items in our sample space decreases by one each time the process is repeated.**

Also at this time it might be a good idea to review some ideas from previous sections that reoccur in the problems:

- Fundamental Principle of Counting – if a process is done in stages, we **multiply** between stages.
- Mutually Exclusive events – if an event can occur in different ways, we count each process individually by the appropriate method and **add** the different ways to obtain the total for the event.
- Diagramming methods – often the problems in this section can be modeled by the methods that were discussed back in Chapter 2. What were these methods and how did we use them? **Tree Diagrams and Slots – for more information refer to page 1 and 2 respectively of this document where they are discussed.**

Special Note: Many of the problems involve using a standard deck of playing cards. Remember there are 52 total cards in a deck, which are divided into four suits of 13 cards – hearts (red), diamonds (red), spades (black), and clubs (black). Each of the 13 cards in the suit has a different rank (value): ace (1), 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, and king - listed in ascending value. **Face cards** are considered to be jacks, queens and kings.

6.4 Examples

Example 1 A single die is tossed twice. What is the probability that the sum of the faces is prime number if a 2 was rolled on the first roll?

Solution Mathematically, our goal is to compute $P(A|B)$ if $A =$"sum of the faces is a prime number" and $B =$"rolling a 2." From the information given, we can say $P(B) = \frac{1}{6}$, which is the probability of rolling a 2.

Our next goal is to compute $P(A \cap B)$ - what is this saying in English? It means "the probability of obtaining a sum that is a prime number and rolling a 2". As we know we rolled a two, there are three prime numbers that we can get: $3(= 2+1)$, $5(= 2+3)$, and $7(= 2+5)$, so we have three outcomes. We also know that the sample space is $6^2 = 36$, which is the total number of outcomes if rolling two dice. Hence,

$$P(A \cap B) = \frac{3}{36} = \frac{1}{12}$$

Now we are able to calculate $P(A|B)$:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}$$

Example 2 A bag of marbles contains 12 blue marbles, 9 red marbles, 8 green marbles, and 7 yellow marbles. Three marbles are drawn in succession without replacement. Find the probability that you draw one blue and two yellow marbles.

Solution The goal of this problem is clear – "find the probability that you draw one blue (B) and two yellow (Y) marbles." We also notice **without replacement** - so our sample space (at the beginning it is 36) will decrease with each draw. Now our goal is to think about the different ways in which we could draw the marbles...

We could draw them in the following different orders: BYY, YBY, and YYB, as the order in which you draw the yellow marbles does not matter as they are identical. So now we compute the probability of these orderings, remembering that these events are mutually exclusive as well as remembering the Fundamental Principle of Counting:

$$P(BYY) + P(YBY) + P(YYB) = \left(\frac{12}{36} * \frac{7}{35} * \frac{6}{34}\right) + \left(\frac{7}{36} * \frac{12}{35} * \frac{6}{34}\right) + \left(\frac{7}{36} * \frac{6}{35} * \frac{12}{34}\right) = \frac{504 + 504 + 504}{42,840} = \frac{3}{85}$$

6.5 Exercises

1. A bag contains 3 blue marbles, 2 red marbles, 7 white marbles, and 3 green marbles. If three marbles are drawn in succession with replacement, what is the probability that at least two are green? *Answer:* $\frac{3^3+3^2(2+7+3)}{15^3} = \frac{1}{25}$ *; Text Problems 27-30, 33-40, 53-56*

2. Let A be the event *drawing a face card from a standard deck of cards* and let B be the event *drawing a heart from a standard deck of cards*. Are the events A and B independent? *Answer: Yes they are independent; Text Problems 15-18*

3. Four marbles are drawn without replacement from a bag containing 2 black, 3 red, 5 green, 7 yellow, 11 orange, and 6 purple marbles. What is the probability that you draw at least 3 orange marbles? *Answer:* $\frac{(11*10*9*8)+[11*10*9](2+3+5+7+6)}{34*33*32*31} = \frac{30,690}{1,113,024} = \frac{15}{544}$ *; Text Problems 23-26, 41-52, 57-60*

4. A survey asked 60 people if they liked bleu cheese and if they liked cottage cheese. After tallying the survey, the researchers found that 19 people liked bleu cheese, 37 liked cottage cheese, and 13 liked both bleu and cottage cheese. What is the probability that one of the people surveyed would like bleu cheese if they like cottage cheese? *Answer:* $\frac{\frac{13}{60}}{\frac{37}{60}} = \frac{13}{37}$ *; Text Problems 1-14*

5. Find the probability of obtaining a flush (five cards from the same suit) from a standard deck of playing cards. Hint: Since the suit is not specified, there are four different cases. Also we are including a royal flush (Ace, King, Queen, Jack, and 10 of the same suit) as well as a straight flush (5 cards in sequence in the same suit). *Answer:* $\frac{C(4,1)C(13,5)}{C(52,5)} = \frac{5148}{2,598,960} = \frac{33}{16,660}$

7 Solutions to Exercises

7.1 Section 2.5

1. Each garden, G, will have a vegetable, V, fertilizer, F, and cages for the vegetables, C. The total number of gardens will depend on the number of different choices for each of these. After counting, we see there are 6 different vegetables, 2 different fertilizers, and 4 different cages. By the Fundamental Principle of Counting, the number of different gardens is

$$G = V * F * C = 6 * 2 * 4 = 48$$

2. This too is a subprocess problem; so the overall goal compute the number of ways for Juan to complete his homework (which is the process as well) and there are 4 subprocesses - the different courses (Calc, Econ, Psych, and Chem). For each subprocess, there are a different number of choices available - 8, 5, 4, and 10 respectively. Since we determined the **Goal, Process, and Subprocess**, this process says we multiply between the number of choices for each (using the Fundamental Principle of Counting) and we obtain that the total is $8*5*4*10 = 1600$.

3. We know that the outcome of the experiment is either a success or a failure (these are the *options* for the outcome of the experiment). As the chemist repeats the experiment 10 times – or has 10 *trials*, by the *options and trials* method discussed in this section, the total number of success-failure strings will be $2^{10} = 1024$.

4. As the hint suggests, the two parts of this problem are superobject type problems. First, combine Analise and Frank into a single object O, so we have 6 object to place and the number of ways to do this is $6*5*4**3*2*1 = 720$ (can be done by slots), but we also have to multiply by the number of ways in which Analise and Frank could sit. There are two ways so our answer is $720*2 = 1440$. The second part is similar, but now we make another object P out of Cathy and Danielle, so we have to place O, P, Ben, Eric, and George in 5 seats. This can be done via slots and it turns out that there are $5*4*3*2*1 = 120$, but now we need to multiply by the number of different arrangements for objects O and P. We know O is 2 and object P also has 2 arrangements. Hence the total is $120*2*2 = 480$.

5. This can be done by slots easily. We have

$$No.1*No.2*No.3*No.4*No.5*No.6*No.7 = Total$$

If no number can be repeated, there are 10 choices for the first number, 9 choices for the second number, etc until there are 4 choice for the seventh number. Hence the Total number of IDs are:

$$10*9*8*7*6*5*4 = 604,800$$

The second part of this problem can be done similarly via slots, except we have more interesting restrictions. For the first number, it must be a number greater than 5 and we have exactly two numbers in the list that are greater than 5 – 8 and 9, so we have 2 options for the first slot. After this, we can place the numbers however we want; there will be 6 options for the second slot, 5 for the third, etc., until there is only 1 option for the seventh slot. By the Fundamental Principle of Counting, the total is

$$2*6*5*4*3*2*1 = 1440$$

7.2 Section 3.5

1. There are two ways to do this: strictly mathematically or make a tree diagram (or equivalent picture) as mentioned in the section. Mathematically, we have 3 total objects and need pick all 3 so we know it is going to be $C(3,3)$ or $P(3,3)$, the difference is in the fact if order matters or not. As the checks are for different amounts, they have *different labels*, meaning that the order matters! (Moreover, if you were supposed to receive the million dollar check and received the one-quarter million dollars check instead, that is a BIG difference!) As order matters, we count with permutations, and so our answer is

$$P(3,3) = 3! = 6$$

2. From the setup of the problem, we are selecting 7 people from the 10 to sit at the table, but we need to figure out to count these via permutations or combinations. Suppose we pick the 7 from the 10. Do the seven have distinct roles or labels associated with them? No - the chairs are not labeled and are considered to be identical, so we count via combinations and get

$$C(10,7) = 120$$

Conversely, we could select the 3 people to sit at a nearby table and the number of ways to do this is

$$C(10,3)$$

since order does not matter once again. Either way we do this, the answer is the same (as it should be)!

3. Using the Multinomial Theorem as the problem suggests, we count the total number of letters in the word (12). Next we identify the different letters in the word and the number of each of them: F (1), A (2), C (1), I (3), L (1), T (2) N (1) G(1). And by the Multinomial Theorem, the number of different words is:

$$\frac{12!}{1!2!1!3!1!2!1!1!} = \frac{12!}{2!3!2!} = 19,958,400$$

4. For the first part, we have 32 total objects and are picking a subset of 12, so we know are "n" and "r", but we need to figure out how to count them. Since we are interested in the *top twelve* teams, our "r" has labels; specifically *the best team, second best team, ..., the twelfth best team.* As it has labels, we count by permutations.

$$P(32,12) = 1.08x10^{17}$$

For the second part, we now have 12 teams and are picking two of them for the top two positions. By the same reasoning as before, we must count by permutations:

$$P(12,2) = 132$$

5. As the candidates for King and Queen cannot take part in the activities, there are 52 students that need to be divided up into each activity. From the given information, we do not know if there are roles to be assigned in either float building or window decorating, and as such we assume there are no roles, meaning we will count via permutations. Additionally, since we only have two activities, once we count the number of ways to choose students for one activity, the rest must go to the other activity. As such, the number of ways to pick students for window decorating is $C(52,25)$, and just to verify, this should be the same as the number of ways to pick students for float building - $C(52,27)$. We see that:

$$C(52,25) = 4.78x10^{14} = C(52,27)$$

For the second portion of the question, it will matter who receives King and Queen, so we count by permutations. We first select one of the four boys for King and then select one of the four girls for Queen. As this is done *in stages*, we use the Fundamental Principle of Counting to obtain the total:

$$P(4,1) * P(4,1) = 4 * 4 = 16$$

For the final part of the question, there are now Prince and Princess positions as well. So we can take our answer from the King-Queen pairings (16) and continue the same process. We have 3 of each gender left and one position for each left. Counting via permutations, this is

$$16 * P(3,1) * P(3,1) = 16 * 3 * 3 = 144$$

7.3 Section 4.5

1. To begin, identify S. Here we have *first, second, and third* places, which are labeled positions, so we count by permutations. There are 6 people and we are picking 3 of them for the positions, so $S = P(6,3) = 120$. Now let's compute E. There is only one way in which Langdon takes first, Nancy takes second, and Katrina takes third as the positions are labeled, so $E = 1$. Hence the probability of this scenario occurring is

$$P = \frac{E}{S} = \frac{1}{120}$$

2. After 10 minutes, the company produces 20 lightbulbs and they are randomly selecting 5, assuming they are all identical. Due to this, we say our $S = C(20,5)$ and we move onto computing E. Also in the time period, 5 of the 20 are known to be defective (meaning 15 are nondefective). According to the constraints of the problem, we want to know what the event is that the sample of 5 will contain 3 defective lightbulbs (and therefore 2 will be nondefective). Again, we assume the lightbulbs are identical (other than the fact that they are either defective or not) and we need to select 3 of the 5 defective ones to be in our event **AND** 2 of the 15 nondefective bulbs as well. Interpreting this mathematically, this is $C(5,3) * C(15,2) = E$ by the Fundamental Principle of Counting. Now we can calculate the Probability:

$$P = \frac{E}{S} = \frac{1,050}{15,504} = \frac{175}{2,584}$$

3. S here is the total number of outcomes after tossing a pair (2) dodecahedral dice (12 sided). This is an Options and trials type problem from the Fundamental Principle of Counting with 12 options and 2 trials. Thus, $S = 12^2 = 144$. Now for E - the number of ways of rolling a sum of 17 with a pair of dodecahedral dice. We see $17 = 12+5 = 5+12 = 11+6 = 6+11 = 10+7 = 7+10 = 9+8 = 8+9$ meaning there are 8 ways to roll a sum of 17, so $E = 8$. As such, the probability is

$$P = \frac{E}{S} = \frac{8}{144} = \frac{1}{18}$$

4. As with all probability problems, we need to compute E and S (the order in which we do so does not matter, as long as they are computed correctly). Fortunately S is somewhat easier to compute; for the fair coin, there are two outcomes or *two options* - heads or tails - and we are repeating the flipping process 10 times, or conducting 10 *trials*. By the Fundamental Principle of Counting (specifically the Options and Trials process), $S = 2^{10} = 1024$. E is more involved to compute. Since we are concerned with “at least 7 heads”, we must figure out the ways to flip 7, 8, 9, and 10 heads. We can count this by combinations as the order in which we flip the heads does not matter here; we are just concerned with the total number of heads. BUT notice we could have 7 OR 8 OR 9 OR 10 heads – meaning we *add* between the possible different outcomes. Hence $E = C(10,7) + C(10,8) + C(10,9) + C(10,10)$. Thus the probability for this problem is:

$$P = \frac{E}{S} = \frac{C(10,7)+C(10,8)+C(10,9)+C(10,10)}{1024} = \frac{11}{64}$$

5. To compute S, we see that this is an Options and Trials problem; there are 5 houses (options) for the 9 letters to be delivered to (trials), and as such, $S = 5^9 = 1,953,125$. Now we must compute E. Since the problem does not mention if the letters are distinguishable (have different labels) or not, we will assume that they are indistinguishable, meaning they do not have different labels meaning we know to count via combinations. Moreover this is a stage process - the mailman will stop at the first house, then the second house, etc. until he stops at the fifth house. But how many letters does he leave at each house? The problem does not tell us other than the fact that we are concerned with computing the number of ways 2 letters can be left at four houses and 1 letter can be left at the other house. Actually, no matter how we compute this number, we will end up with the same number! (Try placing the 1 letter at various places in the following calculation.) With the fact that E is a stage process and the number of letters, we find that $E = C(9,2) * C(7,2) * C(5,2) * C(3,2) * C(1,1) = 22,680$. Also, it should be noted that even though we can place the one letter in various places in the string of calculations, the different arrangements are not *disjoint*, or *mutually exclusive* - see the next section - as we can simply change the order in which we place the houses. (If this argument is unfamiliar to you, check out some textbooks in Enumerative Combinatorics or Discrete Mathematics.) Hence, the probability for this problem is

$$P = \frac{E}{S} = \frac{22,680}{1,953,125} = \frac{4,536}{390,625}$$

7.4 Section 5.4

1. As the problem hint suggests, try using the Complement Rule - so the probability that a 3 or 6 *is not rolled* is the same as 1−the probability that a 3 or 6 *is rolled*. We know that on a standard die, there are 6 total options, meaning $S = 6$ and $E = 2$ as we are interested in rolling a 3 or 6, two of the six sides. As such, the probability of rolling a 3 or 6 is $P = \frac{2}{6} = \frac{1}{3}$. So by the Complement Rule, the probability of not rolling a 3 or 6 is:
$$1 - P(3or6) = 1 - \frac{1}{3} = \frac{2}{3}$$

2. Without incorporating probabilities: Let A denote the set of students working on an experiment and let B denote the set of students working on a worksheet. We know that $A = 18$, $B = 13$ and also, the total number of students in the class means $A \cup B = 25$. As such, we are looking for $A \cap B$. Then using the formula from Elementary Inclusion-Exclusion,
$$A \cap B = A + B - A \cup B = 18 + 13 - 25 = 6$$
Incorporating probabilities (we must do the same procedure as in the first part, but now we must divide by the number of students in the class): $P(A) = \frac{18}{25}$, $P(B) = \frac{13}{25}$, $P(A \cup B) = \frac{25}{25}$. By Elementary Inclusion-Exclusion,
$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{18}{25} + \frac{13}{25} - \frac{25}{25} = \frac{6}{25}$$
As the hint suggests, when we do so, we must multiply our result by 25 and in doing so, we see there are 6 students doing both.

3. Before starting in on the problem, let's do some observing. How man total beads are in the bag? $50 + 30 + 25 + 30 + 15 + 35 + 15 = 200 = S$. Next we notice the **or** - we remember that 'or' means we add! So we add the probability of drawing a white bead and the probability of drawing a blue bead. As there are 35 and 25 white and blue beads respectively, these are our E. Hence
$$P(white) + P(blue) = \frac{E_{white}}{S} + \frac{E_{blue}}{S} = \frac{35}{200} + \frac{25}{200} = \frac{60}{200} = \frac{3}{10} = 0.3$$

4. To do this problem, we try to utilize the hint. First we identify our S - we have 6 options to roll on a die and we roll it 3 times, so by Options and Trials in the Fundamental Principle of Counting, $S = 6^3 = 216$. Before tackling E - remember the hint. By the Complement Rule,

   the probability of a sum greater than 5 = 1 − probability of a sum less than or equal to 5

   and we make the latter probability the E we want to calculate (unless you feel like figuring out over 200 ways to get a sum greater than 5...). As we have three rolls, we can at minimum roll a 3, or we can roll a 4 or 5 as well in this event. We see that we can get 3 in one way: $1+1+1$, 4 in three ways: $2+1+1$, $1+2+1$,$1+1+2$, and 5 in 6 ways: $1+1+3$, $1+3+1$, $3+1+1$, $2+1+2$, $2+2+1$, $1+2+2$ and so our $E = 1+3+6 = 10$. Hence the probability we are looking for is
$$1 - \frac{10}{216} = \frac{206}{216} = \frac{103}{108} = 0.954$$

5. Since we want to figure out if the game is fair or not, want to compute the mathematical expectation and see if it equals *one dollar*, the amount paid for the game. We let x_i be the various values of the prizes:
   - x_1 = the value of prize of 5 dollars = 5
   - x_2 = the value of prize of 10 dollars =10
   - x_3 = the value of prize of 20 dollars = 20

- x_4 = the value of prize of 50 dollars = 50
- x_5 = the value of prize of 100 dollars = 100
- x_6 = the value of prize of 1,000 dollars = 1,000
- x_7 = the value of prize of 5,000 dollars = 5,000

Now we compute the probabilities, p_i, of getting each prize:

- p_1 = the probability of getting the prize of 5 dollars = $\frac{20}{15{,}000} = 1.33x10^{-3}$
- p_2 = the probability of getting the prize of 10 dollars = $\frac{20}{15{,}000} = 1.33x10^{-3}$
- p_3 = the probability of getting the prize of 20 dollars = $\frac{10}{15{,}000} = 6.66x10^{-4}$
- p_4 = the probability of getting the prize of 50 dollars = $\frac{10}{15{,}000} = 6.66x10^{-4}$
- p_5 = the probability of getting the prize of 100 dollars = $\frac{10}{15{,}000} = 6.66x10^{-4}$
- p_6 = the probability of getting the prize of 1,000 dollars = $\frac{8}{15{,}000} = 5.33x10^{-4}$
- p_7 = the probability of getting the prize of 5,000 dollars = $\frac{1}{15{,}000} = 6.66x10^{-5}$

Then the expected value is

$$E_v = x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 + x_5p_5 + x_6p_6 + x_7p_7$$
$$= (5)(1.33x10^{-3}) + (10)(1.33x10^{-3}) + (20)(6.66x10^{-4}) + (50)(6.66x10^{-4}) + (100)(6.66x10^{-4}) +$$
$$+ (1{,}000)(5.33x10^{-4}) + (5{,}000)(6.66x10^{-5}) = 0.99917 = 1.00$$

So the game is fair.

7.5 Section 6.5

1. To begin, we compute S. As the marbles are being drawn with replacement, the sample size does not change each time, meaning for each trial we will still have 15 options and as such, $S = 15^3 = 3375$. Now for our event E. According to the restrictions, we need to count the number of ways we can pull out 2 OR 3 green marbles as we are given the condition 'at least two are green.' So we analyze the overall process in grabs (in steps) and use the Fundamental Principle of Counting! Let's say we are interested in pulling 3 green marbles out. There are 3 green marbles in the bag so each time we go in and grab a marble we could pull out one of the three marbles - so we have 3 options per grab and hence in total we have $3*3*3 = 3^3$ Now consider the case with two green marbles. Our reasoning will be the same as before, but now one of the other marbles is not green so we must consider these events as mutually exclusive - the other marble could be red, white or blue. Hence we also have $3*3*2$, $3*3*7$, and $3*3*3$ and since they are mutually exclusive, we add them, obtaining $3^2(2+7+3)$. As the two processes were linked by an OR, we add the scenarios and obtain $E = 3^3 + 3^2(2+7+3)$. Hence the resulting probability is
$$P = \frac{3^3 + 3^2(2+7+3)}{3375} = \frac{135}{3375} = \frac{1}{25} = 0.04$$

2. These events will be independent if and only if $P(A)P(B) = P(A \cap B)$, as mentioned in the section. We find that $P(A) = \frac{4*3}{52}$ as there are three face cards per suit and there are four suits in a deck of 52 cards, and $P(B) = \frac{13}{52}$ as there are 13 hearts in a deck of cards. As such we now compute $P(A \cap B)$, the probability of drawing a face card that is a heart from a deck of cards and we find this is $P(A \cap B) = \frac{3}{52}$. Now we verify the equation:
$$\frac{12}{52} * \frac{13}{52} = \frac{3}{52}$$
$$\frac{3}{52} = \frac{3}{52}$$
Therefore the events are independent.

3. In total there are 34 marbles in the bag and as we draw without replacement, our sample size decreases by the number drawn (1) each time. Thus our $S = 34 * 33 * 32 * 31 = 1,113,024$. For E we repeat the same process as we did in 1, but this time, our number of orange marbles decreases each time we draw them (as we draw without replacement). If we draw four total marbles and we want at least three to be orange, we are interested in the number of ways to draw 3 OR 4 orange marbles. Let's consider the case where we draw 4 orange marbles (under the constraints of the problem). On the first draw we have 11 choice, on the second we have 10, on the third we have 9 and on the fourth and final draw, we have 8 orange marbles we could draw as we draw without replacement. As this was done in stages, we multiply between the draws and obtain $11 * 10 * 9 * 8$ for one case. Now consider the case where 3 of the four are orange marbles: we know 3 will be orange, giving $11 * 10 * 9$ and the fourth will be of a different color, but as the colors are mutually exclusive events, we must add the different possibilities and in doing so obtain $11 * 10 * 9 * (2+3+5+7+6)$. Hence our $E = 11 * 10 * 9 * 8 + 11 * 10 * 9 * (2+3+5+7+6)$ and the probability is:
$$P = \frac{11 * 10 * 9 * 8 + 11 * 10 * 9 * (2+3+5+7+6)}{1,113,024} = \frac{15}{544}$$

4. Here we are looking for a conditional probability. Let $A =$ the people that like bleu cheese and let $B =$ the people that like cottage cheese. So our conditional probability that we are looking for is $P(A|B)$ and to calculate it, we need to find $P(B)$ and $P(A \cap B)$. $P(B) = \frac{37}{60}$ and $P(A \cap B) = \frac{13}{60}$ by the given information. Hence,
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{13}{60}}{\frac{37}{60}} = \frac{13}{37}$$

5. We know in total, we are selecting 5 cards from 52 and the number of ways to do so is our sample space and so $S = C(52,5)$. Now we need to select five cards from the same suit and to do this, we break it into stages (remember the Fundamental Principle of Counting!): first select a suit and then select the rank, or value, of the 5 cards. To select a suit, there are 4 options, but specifically, this number is $C(4,1)$. Then we pick 5 cards from that suit and the number of ways to do this is $C(13,5)$ as there are 13 cards per suit and we are selecting 5. Hence the total number of ways to get a flush is $E = C(4,1)C(13,5)$ and the probability of getting a flush is
$$P = \frac{C(4,1)C(13,5)}{C(52,5)} = \frac{5,148}{2,598,960} = \frac{33}{16,660} = 1.98x10^{-3}$$