Asymptotic Expansions for Products of Characteristic Functions Under Moment Assumptions of Non-integer Orders

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Abstract This is mostly a review of results and proofs related to asymptotic expansions for characteristic functions of sums of independent random variables (known also as Edgeworth-type expansions). A number of known results is refined in terms of Lyapunov coefficients of non-integer orders.

Let $X_1, \dots, X_n$ be independent random variables with zero means, variances $\sigma_k^2 = \mathrm{Var}(X_k)$, such that $\sum_{k=1}^n \sigma_k^2 = 1$, and with finite absolute moments of some integer order $s \geq 2$. Introduce the Lyapunov coefficients

$$L_s = \sum_{k=1}^n \mathbb{E}\,|X_k|^s \qquad (s \geq 2).$$

If L_3 is small, the distribution F_n of the sum $S_n = X_1 + \dots + X_n$ will be close in a weak sense to the standard normal law with density and distribution function

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad \Phi(x) = \int_{-\infty}^x \varphi(y)\,dy \qquad (x \in \mathbb{R}).$$

This variant of the central limit theorem may be quantified by virtue of the classical Berry-Esseen bound

$$\sup_x\, |\,\mathbb{P}\{S_n \leq x\} - \Phi(x)| \leq cL_3$$

(where c is an absolute constant). Moreover, in case $s > 3$, in some sense the rate of approximation of F_n can be made much better – to be of order at most L_s, if we replace the normal law by a certain "corrected normal" signed measure μ_{s-1} on the real line. The density φ_{s-1} of this measure involves the cumulants γ_p of S_n of orders up to $s-1$ (which are just the sums of the cumulants of X_k); for example,

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$$\varphi_3(x) = \varphi(x)\Big(1 + \frac{\gamma_3}{3!}\,H_3(x)\Big),$$

$$\varphi_4(x) = \varphi(x)\Big(1 + \frac{\gamma_3}{3!}\,H_3(x) + \frac{\gamma_4}{4!}\,H_4(x) + \frac{\gamma_3^2}{2!\,3!^2}\,H_6(x)\Big),$$

where H_k denotes the Chebyshev-Hermite polynomial of degree k. More generally,

$$\varphi_{s-1}(x) = \varphi(x)\,\sum \frac{1}{k_1!\dots k_{s-3}!}\,\Big(\frac{\gamma_3}{3!}\Big)^{k_1}\dots\Big(\frac{\gamma_{s-1}}{(s-1)!}\Big)^{k_{s-3}}\,H_k(x), \tag{0.1}$$

where $k = 3k_1 + \dots + (s-1)k_{s-3}$ and where the summation is running over all collections of non-negative integers $k_1, \dots, k_{s-3}$ such that $k_1 + 2k_2 + \dots + (s-3)k_{s-3} \leq s-3$.

When the random variables $X_k = \frac{1}{\sqrt{n}}\,\xi_k$ are identically distributed, the sum in (0.1) represents a polynomial in $\frac{1}{\sqrt{n}}$ of degree at most $s-3$ with free term 1. In that case, the Lyapunov coefficient

$$L_s = \mathbb{E}\,|\xi_1|^s\,n^{-\frac{s-2}{2}}$$

has a smaller order for growing n in comparison with all terms of the sum.

The closeness of the measures F_n and μ_{s-1} is usually studied with the help of Fourier methods. That is, as the first step, it is established that on a relatively long interval $|t| \leq T$ the characteristic function $f_n(t) = \mathbb{E}\,e^{itS_n}$ together with its first s derivatives are properly approximated by the Fourier-Stieltjes transform

$$g_{s-1}(t) = \int_{-\infty}^{\infty} e^{itx}\,d\mu_{s-1}(x)$$

and its derivatives. In particular, it is aimed to achieve relations such as

$$\big|f_n^{(p)}(t) - g_{s-1}^{(p)}(t)\big| \leq C_s L_s \min\{1, |t|^{s-p}\}\,e^{-ct^2}, \qquad p = 0, 1, \dots, s, \tag{0.2}$$

in which case one may speak about an asymptotic expansion for f_n by means of g_{s-1}. When it turns out possible to convert these relations to the statements about the closeness of the distribution function associated to F_n and μ_{s-1}, one obtains an Edgeworth expansion for F_n (or for density of F_n, when it exists). Basic results in this direction were developed by many researchers in the 1930–1970s, including Cramér, Esseen, Gnedenko, Petrov, Statulevičius, Bikjalis, Bhattacharya and Ranga Rao, Götze and Hipp among others (cf. [C, E, G1, G-K, P1, P2, P3, St1, St2, Bi1, Bi2, Bi3, B-C-G1, B-C-G2, B-C-G3, Pr1, Pr2, Bi1, Bi2, B-RR, G-H, Se, B1]).

In these notes, we focus on the questions that are only related to the first part of the problem, i.e., to the asymptotic expansions for f_n. We review several results, clarify basic technical ingredients of the proofs, and make some refinements where possible. In particular, the following questions are addressed: On which intervals do we have asymptotic expansions for the characteristic functions? How may the constants C_s depend on the growing parameter s? Another issue, which is well motivated, e.g., by limit problems about the normal approximation in terms of transport distances (cf. [B2]), is how to extend corresponding statements to the case of non-integer (or, fractional) values of s.

In a separate (first) part, we collect several results about the distributions of single random variables, including general inequalities on the moments, cumulants, and derivatives of characteristic functions, which lead to corresponding Taylor's expansions. In the second part, there have been collected some results on the behavior of Lyapunov's coefficients and moment inequalities for sums of independent random variables, with first applications to products of characteristic functions. Asymptotic expansions g_{s-1} for f_n are constructed and studied in the third part. In particular, in the interval $|t| \leq cL_s^{-1/3(s-2)}$ (in case L_s is small), we derive a sharper form of (0.2),

$$\left|f_n^{(p)}(t) - g_{s-1}^{(p)}(t)\right| \;\leq\; C^s L_s \max\{|t|^{s-p}, |t|^{3(s-2)+p}\}\, e^{-t^2/2}.$$

This interval of approximation, which we call moderate, appears in a natural way in many investigations, mostly focused on the case $p = 0$ and when X_k's are equidistributed. The fourth part is devoted to the extension of this interval to the size $|t| \leq 1/L_3$ which we call a long interval. This is possible at the expense of the constant in the exponent and with a different behavior of s-dependent factors, by showing that both $f_n^{(p)}(t)$ and $g_{s-1}^{(p)}(t)$ are small in absolute value outside the moderate interval. All results are developed for real values of the main parameter s. More precisely, we use the following plan.

1 Generalized Chain Rule Formula

The following calculus formula is frequently used in a multiple differentiation.

Proposition 1.1 *Suppose that a complex-valued function $y = y(t)$ is defined and has p derivatives in some open interval of the real line ($p \geq 1$). If $z = z(y)$ is analytic in the region containing all values of y, then*

$$\frac{d^p}{dt^p}\, z(y(t)) = p! \sum \frac{d^{s_p}\, z(y)}{dy^{s_p}}\Big|_{y=y(t)} \prod_{r=1}^{p} \frac{1}{k_r!} \bigg(\frac{1}{r!}\, \frac{d^r y(t)}{dt^r}\bigg)^{k_r}, \tag{1.1}$$

where $s_p = k_1 + \cdots + k_p$ and where the summation is performed over all non-negative integer solutions $(k_1, \dots, k_p)$ to the equation $k_1 + 2k_2 + \cdots + pk_p = p$.

This formula can be used to develop a number of interesting identities and inequalities like the following ones given in the next lemma.

Lemma 1.2 *With the summation as before, for any $\lambda \in \mathbb{R}$ and any integer $p \geq 1$,*

$$\sum (s_p - 1)! \prod_{r=1}^{p} \frac{1}{k_r!}\, \lambda^{k_r} \ = \ \frac{(1+\lambda)^p - 1}{p} \tag{1.2}$$

$$\sum s_p! \prod_{r=1}^{p} \frac{1}{k_r!}\, \lambda^{k_r} \ = \ \lambda\, (1+\lambda)^{p-1}. \tag{1.3}$$

In particular, if $0 \leq \lambda \leq 2^{-p}\, \lambda_0$, then

$$\sum \prod_{r=1}^{p} \frac{1}{k_r!}\, \lambda^{k_r} \ \leq \ \lambda\, e^{\lambda_0/4}. \tag{1.4}$$

In addition,

$$\sum \prod_{r=1}^{p} \frac{1}{k_r!} \Big(\frac{\lambda^r}{r}\Big)^{k_r} \ = \ \lambda^p. \tag{1.5}$$

Proof First, apply Proposition 1.1 with $z(y) = -\log(1-y)$, in which case (1.1) becomes

$$-\frac{d^p}{dt^p}\log(1-y(t)) = p!\sum \frac{(s_p-1)!}{(1-y(t))^{s_p}}\prod_{r=1}^{p}\frac{1}{k_r!}\Big(\frac{1}{r!}\,\frac{d^r y(t)}{dt^r}\Big)^{k_r}. \tag{1.6}$$

Choosing $y(t) = \lambda\,\frac{t}{1-t} = -\lambda + \lambda(1-t)^{-1}$ so that $\frac{d^r y(t)}{dt^r} = r!\,\lambda(1-t)^{-(r+1)}$, the above sum on the right-hand side equals

$$\sum (s_p-1)!\,(1-y(t))^{-s_p}\,(1-t)^{-p-s_p}\lambda^{s_p}\prod_{r=1}^{p}\frac{1}{k_r!}.$$

On the other hand, writing $-\log(1-y(t)) = \log(1-t) - \log\big((1+\lambda)(1-t)-\lambda\big)$, we get

$$-\frac{d^p}{dt^p}\log(1-y(t)) = -\frac{(p-1)!}{(1-t)^p} + (1+\lambda)^p\,\frac{(p-1)!}{\big((1+\lambda)(1-t)-\lambda\big)^p}.$$

Therefore, (1.6) yields

$$(p-1)!\bigg[\frac{(1+\lambda)^p}{\big((1+\lambda)(1-t)-\lambda\big)^p} - \frac{1}{(1-t)^p}\bigg] = p!\sum\frac{(s_p-1)!\,\lambda^{s_p}}{(1-y(t))^{s_p}\,(1-t)^{p+s_p}}\prod_{r=1}^{p}\frac{1}{k_r!}.$$

Putting $t=0$, we obtain the identity (1.2). Differentiating it with respect to λ and multiplying by λ, we arrive at (1.3). In turn, using $s_p! \geq 1$ and the property that the function $p \to (p-1)2^{-p}$ is decreasing in $p \geq 2$, (1.3) implies that, for all $p \geq 2$,

$$(1+\lambda)^{p-1} \leq e^{(p-1)\lambda} \leq e^{\lambda_0(p-1)2^{-p}} \leq e^{\lambda_0/4},$$

which obviously holds for $p=1$ as well.

Finally, let us apply (1.1) with $z(y) = e^y$, when this identity becomes

$$\frac{d^p}{dt^p}\,e^{y(t)} = p!\,e^{y(t)}\sum\prod_{r=1}^{p}\frac{1}{k_r!}\Big(\frac{1}{r!}\,\frac{d^r y(t)}{dt^r}\Big)^{k_r}. \tag{1.7}$$

It remains to choose here $y(t) = -\log(1-\lambda t)$, so that $\frac{d^r y(t)}{dt^r} = \lambda^r\,(r-1)!\,(1-t)^{-r}$, and then this equality yields (1.5) at the point $t=0$. □

For an illustration, consider Gaussian functions $g(t) = e^{-t^2/2}$. By the definition,

$$g^{(p)}(t) = (-1)^{p-1}H_p(t)g(t),$$

where H_p denotes the Chebyshev–Hermite polynomial of degree p with leading term 1. From (1.7) with $y(t) = -t^2/2$, we have

$$g^{(p)}(t) = p!\, g(t) \sum_{k_1+2k_2=p} \frac{(-t)^{k_1}}{k_1!k_2!}\, 2^{-k_2}.$$

Using $|t|^{k_1} \leq \max\{1, |t|^p\}$ and applying the identity (1.5), we get a simple upper bound

$$|H_p(t)| \leq p! \max\{1, |t|^p\}. \tag{1.8}$$

2 Logarithm of the Characteristic Functions

If a random variable X has finite absolute moment $\beta_p = \mathbb{E}\,|X|^p$ for some integer $p \geq 1$, its characteristic function $f(t) = \mathbb{E}\, e^{itX}$ has continuous derivatives up to order p and is non-vanishing in some interval $|t| \leq t_0$. Hence, in this interval the principal value of the logarithm $\log f(t)$ is well defined and also has continuous derivatives up to order p, which actually can be expressed explicitly in terms of the first derivatives of f. More precisely, the chain rule formula of Proposition 1.1 with $z(y) = \log y$ immediately yields the following identity:

Proposition 2.1 *Let $\beta_p < \infty$ $(p \geq 1)$. In the interval $|t| \leq t_0$, where $f(t)$ is non-vanishing,*

$$\frac{d^p}{dt^p} \log f(t) = p! \sum \frac{(-1)^{s_p-1}\,(s_p-1)!}{f(t)^{s_p}} \prod_{r=1}^{p} \frac{1}{k_r!} \Big(\frac{1}{r!} f^{(r)}(t)\Big)^{k_r}, \tag{2.1}$$

where $s_p = k_1 + \cdots + k_p$ and the summation is running over all tuples $(k_1, \dots, k_p)$ of non-negative integers such that $k_1 + 2k_2 + \cdots + pk_p = p$.

As was shown by Sakovič [Sa], in the interval $\sqrt{\beta_2}\,|t| \leq \frac{\pi}{2}$ we necessarily have $\mathrm{Re}(f(t)) \geq 0$. This result was sharpened by Rossberg [G2] proving that

$$\mathrm{Re}(f(t)) \geq \cos(\sqrt{\beta_2}\,|t|) \quad \text{for} \quad \sqrt{\beta_2}\,|t| \leq \pi.$$

See also Shevtsova [Sh2] for a more detailed exposition of the question. Thus, the representation (2.1) holds true in the open interval $\sqrt{\beta_2}\,|t| < \frac{\pi}{2}$.

To quickly see that $f(t)$ is non-vanishing on a slightly smaller interval, one can just apply Taylor's formula. Indeed, if $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \beta_2 = \sigma^2$ $(0 < \sigma < \infty)$, then $f(0) = 1, f'(0) = 0$, $|f''(t)| \leq \sigma^2$, and we get

$$|1 - f(t)| \leq \sup_{|z| \leq |t|} |f''(z)|\, \frac{t^2}{2} \leq \frac{\sigma^2 t^2}{2} < 1$$

for $\sigma|t| < \sqrt{2}$. In particular, $|f(t)| \geq \frac{1}{2}$ for $\sigma|t| \leq 1$, so that in this interval the principal value of the logarithm $\log f(t)$ is continuous and has continuous derivatives up to order p.

Let us mention several particular cases in (2.1). Clearly, $(\log f)' = f'f^{-1}$ and $(\log f)'' = f''f^{-1} - f'^2 f^{-2}$. The latter formula can be given in an equivalent form.

Proposition 2.2 *If the variance* $\sigma^2 = \mathrm{Var}(X)$ *is finite, then at any point t such that* $f(t) \neq 0$, *we have*

$$(\log f(t))'' = -\frac{1}{2f(t)^2}\,\mathbb{E}\,(X-Y)^2\, e^{it(X+Y)},$$

where Y is an independent copy of X. In particular,

$$|(\log f(t))''| \leq \frac{\sigma^2}{|f(t)|^2}. \tag{2.2}$$

Indeed, the right-hand side of the equality $f(t)^2\,(\log f(t))'' = f''(t)f(t) - f'(t)^2$ may be written as

$$\begin{aligned} -\big(\,\mathbb{E}X^2 e^{it(X+Y)} - \mathbb{E}\,XY\, e^{it(X+Y)}\big) &= -\Big(\,\mathbb{E}\,\frac{X^2+Y^2}{2}\; e^{it(X+Y)} - \mathbb{E}\,XY\, e^{it(X+Y)}\Big) \\ &= -\frac{1}{2}\,\mathbb{E}\,(X-Y)^2\, e^{it(X+Y)}. \end{aligned}$$

Therefore,

$$|f(t)|^2\,|(\log f(t))''| \leq \frac{1}{2}\,\mathbb{E}\,(X-Y)^2 = \mathrm{Var}(X).$$

For the next two derivatives, let us note that

$$f(t)^3\,(\log f(t))''' = f'''(t)f(t)^2 - 3f''(t)f'(t)f(t) + 2f'(t)^3, \tag{2.3}$$

$$\begin{aligned} f(t)^4\,(\log f(t))'''' = f''''(t)f(t)^3 &- 4f'''(t)f'(t)f(t)^2 - 3f''(t)^2 f(t)^2 \\ &+ 12f''(t)f'(t)^2 f(t) - 6f'(t)^4. \end{aligned} \tag{2.4}$$

3 Moments and Cumulants

Again, let a random variable X have a finite absolute moment $\beta_p = \mathbb{E}\,|X|^p$ for an integer $p \geq 1$. Since the characteristic function $f(t) = \mathbb{E}\, e^{itX}$ is non-vanishing in some interval $|t| \leq t_0$, and $\log f(t)$ has continuous derivatives up to order p, one may introduce the normalized derivatives at zero

$$\gamma_r = \gamma_r(X) = \frac{d^r}{i^p\,dt^r}\,\log f(t)\big|_{t=0}, \qquad r = 0, 1, 2, \dots, p,$$

called the cumulants of X. Each γ_p is determined by the first moments $\alpha_r = \mathbb{E}X^r$, $r = 1, \dots, p$. Namely, at $t = 0$, the identity (2.1) of Proposition 1.1 gives:

Proposition 3.1 *Let $\beta_p < \infty$ $(p \geq 1)$. For $|t| \leq t_0$, we have*

$$\gamma_p = p! \sum (-1)^{s_p - 1}\,(s_p - 1)! \prod_{r=1}^{p} \frac{1}{k_r!}\Big(\frac{\alpha_r}{r!}\Big)^{k_r}, \tag{3.1}$$

where $s_p = k_1 + \dots + k_p$ and where the summation is running over all tuples $(k_1, \dots, k_p)$ of non-negative integers such that $k_1 + 2k_2 + \dots + pk_p = p$.

For example, $\gamma_1 = \alpha_1$, $\gamma_2 = \alpha_2 - \alpha_1^2$. Moreover, if $\alpha_1 = \mathbb{E}X = 0$, $\sigma^2 = \mathbb{E}X^2$, then

$$\gamma_1 = \alpha_1, \quad \gamma_2 = \alpha_2 = \sigma^2, \quad \gamma_3 = \alpha_3, \quad \gamma_4 = \alpha_4 - 3\alpha_2^2 = \beta_4 - 3\sigma^4.$$

One may reverse (3.1) by applying the generalized chain rule to the composition $f(t) = e^{\log f(t)}$, see (1.7). We then get a similar formula

$$\alpha_p = p! \sum \prod_{r=1}^{p} \frac{1}{k_r!}\Big(\frac{\gamma_r}{r!}\Big)^{k_r}. \tag{3.2}$$

Let us now turn to the question of bounding the cumulants in terms of the moments. By Markov's inequality, there are uniform bounds on the derivatives $|f^{(r)}(t)| \leq \beta_r \leq \beta_p^{r/p}$ for $r = 1, \dots, p$. Hence, the combination of identity (1.2) of Lemma 1.2 with $\lambda = \frac{1}{|f(t)|}$ and identity (2.1) of Proposition 2.1 leads to the bound

$$\Big|\frac{d^p}{dt^p}\,\log f(t)\Big| \leq \Big[\Big(1 + \frac{1}{|f(t)|}\Big)^p - 1\Big]\,(p-1)!\,\beta_p. \tag{3.3}$$

This inequality may be compared to the result of Bikjalis [Bi3], who showed that

$$\Big|\frac{d^p}{dt^p}\,\log f(t)\Big| \leq \frac{1}{|f(t)|^p}\,2^{p-1}\,(p-1)!\,\beta_p. \tag{3.4}$$

In particular, when $|f(t)| \geq \frac{1}{2}$, it gives the relation $\big|\frac{d^p}{dt^p}\,\log f(t)\big| \leq 2^{2p-1}\,(p-1)!\,\beta_p$. However, in this case (3.3) yields a better bound

$$\Big|\frac{d^p}{dt^p}\,\log f(t)\Big| \leq (3^p - 1)\,(p-1)!\,\beta_p.$$

We will discuss further sharpenings in the next section, and now just note that at the point $t = 0$, (3.4) provides a bound on the cumulants, $|\gamma_p| \leq (2^{p-1} - 1)\,(p-1)!\,\beta_p$. Another result of Bikjalis [Bi3] provides an improvement for mean zero random variables.

Proposition 3.2 *If $\beta_p = \mathbb{E}\,|X|^p < \infty$ for some integer $p \geq 1$, and $\mathbb{E}X = 0$, then*

$$|\gamma_p| \leq (p-1)!\,\beta_p. \tag{3.5}$$

Proof The case $p = 1$ is obvious. Since $\gamma_2 = \alpha_2 = \beta_2$ in case $\mathbb{E}X = 0$, the desired bound also follows for $p = 2$. So, let $p \geq 3$. Differentiating the identity $f'(t) = f(t)\,(\log f(t))'$ near zero $p-1$ times in accordance with the binomial formula, one gets

$$\frac{d^p}{dt^p} f(t) = \sum_{r=0}^{p-1} C_{p-1}^r \, \frac{d^{p-1-r}}{dt^{p-1-r}} f(t)\, \frac{d^{r+1}}{dt^{r+1}} \log f(t),$$

where here and in the sequel we use the notation $C_n^k = \frac{n!}{k!(n-k)!}$ for the binomial coefficients. Equivalently,

$$\frac{d^p}{dt^p} \log f(t) = \frac{1}{f(t)}\, \frac{d^p}{dt^p} f(t) - \frac{1}{f(t)} \sum_{r=0}^{p-2} C_{p-1}^r \, \frac{d^{p-1-r}}{dt^{p-1-r}} f(t)\, \frac{d^{r+1}}{dt^{r+1}} \log f(t). \tag{3.6}$$

At $t = 0$, this identity becomes

$$\gamma_p = \alpha_p - \sum_{r=0}^{p-3} C_{p-1}^r \, \alpha_{p-1-r}\, \gamma_{r+1},$$

where we used the assumption $\alpha_1 = 0$. One can now proceed by induction on p. Since $|\alpha_{p-1-r}| \leq \beta_p^{(p-1-r)/p}$ and $\gamma_{r+1} \leq r!\beta_p^{(r+1)/p}$ (the induction hypothesis), we obtain that

$$|\gamma_p| \leq \beta_p + \beta_p \sum_{r=0}^{p-3} C_{p-1}^r \, r! = (p-1)!\,\beta_p \left[\frac{1}{(p-1)!} + \sum_{r=0}^{p-3} \frac{1}{(p-1-r)!}\right].$$

The expression in the square brackets $\frac{1}{(p-1)!} + (\frac{1}{2!} + \cdots + \frac{1}{(p-1)!})$ is equal to 1 for $p = 2$ and is smaller than $\frac{1}{6} + (e-2) < 1$ for $p \geq 3$. □

The factorial growth of the constant in the inequality (3.5) is optimal, up to an exponentially growing factor, which was noticed by Bulinskii [Bu] in his study of upper bounds in a more general scheme of random vectors and associated mixed cumulants. To illustrate possible lower bounds, he considered the symmetric Bernoulli distribution assigning the mass $\frac{1}{2}$ to the points ± 1. In this case, the characteristic function is $f(t) = \cos t$, and one may use the Taylor expansion

$$\log f(t) = \log\cos t = -\sum_{p=1}^{\infty} \frac{2^{2p}\,(2^{2p}-1)}{(2p)!}\, B_p\, \frac{t^{2p}}{2p}, \qquad |t| < \frac{\pi}{4},$$

involving Bernoulli numbers $B_p = \frac{2\,(2p)!}{(2\pi)^{2p}}\, d_{2p}$, where $d_{2p} = \sum_{n=1}^{\infty} \frac{1}{n^{2p}}$. Thus, for even integer values of p,

$$|\gamma_p| = \frac{2^p\,(2^p-1)}{p}\, B_{p/2} = \frac{2\,(2^p-1)}{\pi^p}\,(p-1)!\, d_p.$$

From Stirling's formula, one gets $|\gamma_p| \geq (\frac{p}{\pi e})^p \sqrt{2\pi}$. To compare with the upper bound of Proposition 2.2, note that in this Bernoulli case, $\beta_p = 1$ for all p.

4 Bounds on the Derivatives of the Logarithm

We will now extend the Bikjalis argument, so as to obtain the following improvement of the bounds (3.3)–(3.4), assuming that X has mean zero and t is small enough. More precisely, we are going to derive the bound

$$\Big|\frac{d^p}{dt^p}\, \log f(t)\Big| \;\leq\; (p-1)!\,\beta_p \tag{4.1}$$

in the interval $\sigma|t| \leq \varepsilon = \frac{1}{5}$ (except for the value $p = 2$), where $\sigma^2 = \beta_2 = \mathbb{E}X^2$. This can be done with the help of the lower bound

$$|f(t)| \geq 1 - \frac{\sigma^2 t^2}{2} \geq 1 - \frac{\varepsilon^2}{2}, \qquad \sigma|t| \leq \varepsilon. \tag{4.2}$$

First let us check (4.1) for the first 4 values of p. Since $|f'(t)| \leq \sigma^2|t|$, we have

$$|(\log f(t))'| \leq \frac{\beta_2|t|}{|f(t)|} \leq \frac{0.2\,\beta_2^{1/2}}{1-\frac{\varepsilon^2}{2}} \leq 0.21\beta_2^{1/2}. \tag{4.3}$$

When $p = 2$, according to inequality (2.2) of Proposition 2.2,

$$|(\log f(t))''| \leq \frac{\beta_2}{|f(t)|^2} \leq \frac{\beta_2}{(1-\frac{\varepsilon^2}{2})^2} \leq 1.05\beta_2. \tag{4.4}$$

When $p = 3$, we use (2.3) giving

$$|(\log f(t))'''| \;\leq\; \frac{1+3\varepsilon+2\varepsilon^3}{|f(t)|^3}\,\beta_3 \;\leq\; \frac{1+3\varepsilon+2\varepsilon^3}{(1-\frac{\varepsilon^2}{2})^3}\,\beta_3 \;\leq\; 2\beta_3.$$

When $p = 4$, we use (2.4) giving similarly

$$|(\log f(t))''''| \le \frac{4 + 4\varepsilon + 12\varepsilon^2 + 6\varepsilon^4}{|f(t)|^4}\,\beta_4 \le \frac{4 + 4\varepsilon + 12\varepsilon^2 + 6\varepsilon^4}{(1 - \frac{\varepsilon^2}{2})^4}\,\beta_4 \le 6\beta_4.$$

In order to derive (4.1) for $p \ge 5$, we perform the induction step, applying (4.3)–(4.4) and assuming that, in the interval $\sigma|t| \le \varepsilon$,

$$|(\log f(t))^{(r)}| \le (r-1)!\,\beta_r \qquad \text{for } 3 \le r \le p-1. \tag{4.5}$$

By this hypothesis, using the recursive formula (3.6) and the bounds (4.3)–(4.4), we have

$$\begin{aligned}
|f(t)|\,|(\log f(t))^{(p)}| &\le |f^{(p)}(t)| + \sum_{r=0}^{p-2} C_{p-1}^r\,|f^{(p-1-r)}(t)|\,|(\log f(t))^{(r+1)}| \\
&= |f^{(p)}(t)| + (p-1)\,|f'(t)|\,|(\log f(t))^{(p-1)}| \\
&\quad + \sum_{r=2}^{p-3} C_{p-1}^r\,|f^{(p-1-r)}(t)|\,|(\log f(t))^{(r+1)}| \\
&\quad + |f^{(p-1)}(t)|\,|(\log f(t))'| + (p-1)\,|f^{(p-2)}(t)|\,|(\log f(t))''| \\
&\le \beta_p + (p-1)\,\beta_p^{1/p}\,\varepsilon \cdot \beta_{p-1}(p-2)! \\
&\quad + \sum_{r=2}^{p-3} C_{p-1}^r\,\beta_{p-1-r}\,\beta_{r+1}\,r! + \beta_{p-1} \cdot 0.21\beta_2^{1/2} \\
&\quad + (p-1)\beta_{p-2} \cdot 1.05\,\beta_2.
\end{aligned}$$

Here we apply again $\beta_r \le \beta_p^{r/p}$, giving

$$\begin{aligned}
\frac{|f(t)|}{\beta_p}\,|(\log f(t))^{(p)}| &\le 1 + (p-1)!\left[\varepsilon + \sum_{r=2}^{p-3} \frac{1}{(p-1-r)!}\right] + 0.21 + 1.05\,(p-1) \\
&\le 1 + (p-1)!\left[\varepsilon + \sum_{k=2}^{p-1} \frac{1}{k!}\right] + 0.05\,(p-1) \\
&\le 1 + 0.05\,(p-1) + (p-1)!\,(\varepsilon + e - 2).
\end{aligned}$$

Applying the lower bound (4.2), we obtain that

$$\frac{1}{\beta_p}\,|(\log f(t))^{(p)}| \le \frac{1}{1 - \frac{\varepsilon^2}{2}}\,\big(1 + 0.05\,(p-1) + (p-1)!\,(\varepsilon + e - 2)\big).$$

The latter expression does not exceed $(p-1)!$ (which is needed to make the induction step, i.e., to derive (4.5) for $r = p$), if and only if this is true for $p = 5$ (since after division by $(p-1)!$ the expression on the right will be decreasing in p). That is, we need to verify that $\frac{1}{1-\frac{\varepsilon^2}{2}}\,(1.2 + 24\,(\varepsilon + e - 2)) \leq 24$, which is indeed true for $\varepsilon = 0.2$. Hence, we have proved:

Proposition 4.1 *Let X be a random variable such that $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2$ $(\sigma > 0)$ and $\beta_p = \mathbb{E}\,|X|^p < \infty$ for some integer $p \geq 2$. Then, in the interval $\sigma|t| \leq \frac{1}{5}$, the characteristic function $f(t)$ of X is not vanishing and satisfies*

$$|(\log f(t))'| \leq 0.21\,\sigma, \qquad |(\log f(t))''| \leq 1.05\,\sigma.$$

Moreover, if $p \geq 3$, then

$$\Big|\frac{d^p}{dt^p}\,\log f(t)\Big| \,\leq\, (p-1)!\,\beta_p.$$

5 Taylor Expansion for Fourier-Stieltjes Transforms

Let X be a random variable with finite absolute moment $\beta_s = \mathbb{E}\,|X|^s$ of a real order $s > 0$, not necessarily integer. Put

$$\mathbb{E}X^k = \alpha_k, \quad \mathbb{E}\,|X|^k = \beta_k \quad (k = 0, 1, \dots, [s]).$$

In general, suitable expansions for the characteristic function $f(t) = \mathbb{E}\,e^{itX}$ can be developed according to the Taylor formula. Since f has $[s]$ continuous derivatives with $f^{(k)}(0) = i^k\alpha_k$, it admits the Taylor expansion

$$f(t) = \sum_{k=0}^{m} \alpha_k\,\frac{(it)^k}{k!} + \delta(t) \tag{5.1}$$

with $\delta(t) = o(t^m)$, where here and elsewhere we represent $s = m + \alpha$ with integer m and $0 < \alpha \leq 1$. The remainder term can be bounded in terms of β_s as follows:

Proposition 5.1 *For all t,*

$$\Big|\frac{d^p}{dt^p}\,\delta(t)\Big| \,\leq\, 2\beta_s\,\frac{|t|^{s-p}}{(m-p)!}, \qquad p = 0, 1, \dots, m. \tag{5.2}$$

Moreover, if $s = m + 1$ is integer, then

$$\Big|\frac{d^p}{dt^p}\,\delta(t)\Big| \,\leq\, \beta_s\,\frac{|t|^{s-p}}{(s-p)!}, \qquad p = 0, 1, \dots, s.$$

Proof By the very definition, $\delta(t) = \mathbb{E}\, R_m(tX)$, where $R_m(u) = e^{iu} - \sum_{l=0}^{m} \frac{(iu)^l}{l!}$, so that

$$\delta^{(p)}(t) = \mathbb{E}\,(iX)^p\, R_{m-p}(tX).$$

Given an integer number $k \geq 1$, note that $R_k^{(j)}(0) = 0$ for all $j = 0, \dots, k$ with $|R_k^{(k+1)}(u)| = 1$. In addition, $R_k^{(k)}(u) = i^k(e^{iu} - 1)$, so that $|R_k^{(k)}(u)| \leq 2$. Hence, by Taylor's formula,

$$|R_k(u)| \leq \frac{|u|^{k+1}}{(k+1)!} \qquad \text{and} \qquad |R_k(u)| \leq 2\,\frac{|u|^k}{k!}.$$

Although some other interesting bounds on the functions R_k are available (cf., e.g., [Sh1]), these two inequalities are sufficient to conclude that, for any $\alpha \in [0, 1]$,

$$\begin{aligned} |R_k(u)| &\leq \min\left\{2\,\frac{|u|^k}{k!},\, \frac{|u|^{k+1}}{(k+1)!}\right\} \\ &= \frac{|u|^k}{k!}\, \min\left\{2,\, \frac{|u|}{k+1}\right\} \;\leq\; \frac{|u|^k}{k!} \cdot \frac{2^{1-\alpha}}{(k+1)^\alpha}\, |u|^\alpha \;\leq\; \frac{2|u|^{k+\alpha}}{k!}\,. \end{aligned}$$

Therefore,

$$|\delta^{(p)}(t)| \leq \mathbb{E}\left[|X|^p\, \frac{2\,|tX|^{(m-p)+\alpha}}{(m-p)!}\right] = \frac{2\,|t|^{(s-p)}}{(m-p)!}\,\beta_s.$$

In case $s = m + 1$, the function $w(t) = \delta^{(p)}(t)$ has zero derivatives at $t = 0$ up to order $s-p-1$, while $w^{(s-p)}(t) = \delta^{(s)}(t) = \mathbb{E}\,(iX)^s e^{itX}$ is bounded in absolute value by β_s. Hence, by Taylor's formula,

$$|w(t)| \;\leq\; \max_{|z|\leq|t|} |w^{(s-p)}(z)|\, \frac{|t|^{s-p}}{(s-p)!} \;\leq\; \beta_s\, \frac{|t|^{s-p}}{(s-p)!}.$$

□

More generally, consider the Fourier-Stieltjes transform $a(t) = \int_{-\infty}^{\infty} e^{itx}\, d\mu(x)$ of a Borel signed measure μ on the real line and introduce the corresponding absolute moment

$$\beta_s(\mu) = \int_{-\infty}^{\infty} |x|^s\, |\mu(dx)|,$$

where $|\mu|$ is the variation of μ treated as a positive measure on the line, and $s > 0$ is a real number. Clearly, a is $[s]$ times continuously differentiable on $\mathbb{R}$ with derivatives at the origin

$$a^{(p)}(0) = \int_{-\infty}^{\infty} (ix)^p \, d\mu(x), \qquad p = 0, 1, \dots, [s].$$

Here is a natural generalization of Proposition 5.1.

Proposition 5.2 *Let $s = m + \alpha$ with $m \geq 0$ integer and $0 < \alpha \leq 1$. If $a^{(p)}(0) = 0$ for all $p = 0, 1, \dots, m$, then for all $t \in \mathbb{R}$,*

$$|a^{(p)}(t)| \leq 2\beta_s(\mu) \, \frac{|t|^{s-p}}{(m-p)!}, \qquad p = 0, 1, \dots, m.$$

Moreover, if $s = m + 1$ is integer, then

$$|a^{(p)}(t)| \leq \beta_s(\mu) \, \frac{|t|^{s-p}}{(s-p)!}, \qquad p = 0, 1, \dots, [s].$$

Proof Note that $\mu(\mathbb{R}) = 0$ due to $a(0) = 0$. To prove the statement, one can repeat the arguments used in the proof of Proposition 5.1. By the moment assumption, $a(t) = \int_{-\infty}^{\infty} R_m(tx) \, d\mu(x)$, so

$$a^{(p)}(t) = \int_{-\infty}^{\infty} (ix)^p \, R_{m-p}(tx) \, d\mu(x).$$

Using the previous bound $|R_k(u)| \leq \frac{2|u|^{k+\alpha}}{k!}$ with $k = m - p$, we conclude that

$$|a^{(p)}(t)| \leq \int_{-\infty}^{\infty} \left[|x|^p \, \frac{2\,|tx|^{(m-p)+\alpha}}{(m-p)!} \right] |d\mu(x)| = \frac{2\,|t|^{(s-p)}}{(m-p)!} \, \beta_s(\mu).$$

The case $s = m + 1$ is similar. □

6 Taylor Expansion for Logarithm of Characteristic Functions

Our next task is to develop the Taylor expansion for $\log f(t)$ in analogy with the expansion (5.1) for the characteristic function $f(t)$ with a bound similar to (5.2), which would hold even if t is close to zero. Note that, in the most important case $p = m$, that bound yields

$$|f^{(m)}(t) - i^m \alpha_m| \;\leq\; 2\beta_s \, |t|^{\alpha}. \tag{6.1}$$

Hence, we need to derive a similar bound for $\log f(t)$, by replacing α_m with the cumulant γ_m.

We keep the same assumption as in the previous section: $\mathbb{E}X = 0$, $\beta_s = \mathbb{E}\,|X|^s < \infty$, $s = m + \alpha$ with $m \geq 2$ integer and $0 < \alpha \leq 1$. Let us return to the recursive formula

$$f(t)\,(\log f(t))^{(m)} = f^{(m)}(t) - \sum_{r=1}^{m-1} C_{m-1}^{r-1} f^{(m-r)}(t)\,(\log f(t))^{(r)}, \tag{6.2}$$

which at $t = 0$ becomes

$$i^m \gamma_m = i^m \alpha_m - \sum_{r=1}^{m-1} C_{m-1}^{r-1}\, i^{m-r} \alpha_{m-r}\, i^r \gamma_r. \tag{6.3}$$

Since $\alpha_1 = \gamma_1 = 0$, the last summation may be reduced to the values $2 \leq r \leq m-2$ for $m \geq 4$, while there is no sum for $m = 3$.

To argue by induction on m, our induction hypothesis will be

$$|(\log f(t))^{(r)} - i^r \gamma_r| \leq AB^r (r-1)!\, \beta_{r+\alpha} |t|^\alpha, \qquad r = 1, 2, \dots, m-1, \tag{6.4}$$

in the interval $\sigma|t| \leq \frac{1}{5}$, where the parameters $A, B \geq 1$ are to be chosen later on. Recall that Proposition 4.1 provides in this interval the bound

$$|(\log f(t))^{(r)}| \leq A_r (r-1)!\, \beta_r, \qquad r = 2, \dots, m, \tag{6.5}$$

with constants $A_2 = 1.05$ and $A_r = 1$ for $r \geq 3$. Now, let us apply (6.1) with $s = (m-r) + \alpha$. Then we have a similar relation

$$|f^{(m-r)}(t) - i^{m-r} \alpha_{m-r}| \leq 2\beta_{m-r+\alpha} |t|^\alpha, \qquad r = 0, 1, \dots, m-1, \tag{6.6}$$

which is valid for all t. Write

$$\begin{aligned} f^{(m-r)}(t)\,(\log f(t))^{(r)} &= (f^{(m-r)}(t) - i^{m-r}\alpha_{m-r})\,(\log f(t))^{(r)} \\ &\quad + i^{m-r}\alpha_{m-r}\,\big(\log f(t))^{(r)} - i^r \gamma_r\big) + i^{m-r}\alpha_{m-r}\, i^r \gamma_r. \end{aligned}$$

Applying the bounds (6.4)–(6.6) for $r = 2, \dots, m-1$, we get

$$\begin{aligned} |f^{(m-r)}(t)\,(\log f(t))^{(r)} - i^{m-r}\alpha_{m-r}\, i^r \gamma_r| &\leq 2\beta_{m-r+\alpha}|t|^\alpha \cdot A_r (r-1)!\, \beta_r \\ &\quad + \beta_{m-r} \cdot AB^r (r-1)!\, \beta_{r+\alpha} |t|^\alpha \\ &\leq (r-1)!\, \beta_s |t|^\alpha\, (2A_r + AB^r). \end{aligned}$$

When $r = 1$, we use a different bound based on the assumption that $\alpha_1 = \gamma_1 = 0$. Namely, by Proposition 4.1 in part concerning the first derivative, we have

$$|f^{(m-1)}(t)\,(\log f(t))'| \leq 2\beta_{m-1+\alpha} |t|^\alpha \cdot A_1 \beta_2^{1/2} \leq 2A_1\, \beta_s |t|^\alpha,$$

where $A_1 = 0.21$. Hence, subtracting the representation (6.3) from (6.2) and applying the bound (6.1), we get

$$|f(t)\,(\log f(t))^{(m)} - i^m\gamma_m| \le 2\beta_s|t|^\alpha + (m-1)!\,\beta_s|t|^\alpha \sum_{r=1}^{m-1}\frac{1}{(m-r)!}\,(2A_r + AB^r)$$

$$= AB^m\,(m-1)!\,\beta_s|t|^\alpha\Big[\frac{2}{AB^m} + \sum_{k=1}^{m-1}\frac{1}{k!}\Big(\frac{2A_{m-k}}{AB^m} + B^{-k}\Big)\Big].$$

In addition, since $|f(t)-1| \le 2\beta_\alpha|t|^\alpha$, we have

$$|f(t)\,(\log f(t))^{(m)} - (\log f(t))^{(m)}| \le A_m(m-1)!\,\beta_m\cdot 2\beta_\alpha|t|^\alpha \le 2(m-1)!\,\beta_s|t|^\alpha.$$

Hence

$$|(\log f(t))^{(m)} - i^m\gamma_m| \le AB^m\,(m-1)!\,\beta_s|t|^\alpha\Big[\frac{4}{AB^m} + \sum_{k=1}^{m-1}\frac{1}{k!}\Big(\frac{2A_{m-k}}{AB^m} + B^{-k}\Big)\Big],$$

and we can make an induction step by proving (6.4) for $r = m$, once the parameters satisfy

$$\frac{4}{AB^m} + \sum_{k=1}^{m-1}\frac{1}{k!}\Big(\frac{2A_{m-k}}{AB^m} + B^{-k}\Big) \le 1.$$

To simplify, let us use a uniform bound $A_{m-k} \le 1.05$, so that to estimate the above left-hand side from above by

$$\frac{4}{AB^m} + \sum_{k=1}^{\infty}\frac{1}{k!}\Big(\frac{2.1}{AB^m} + B^{-k}\Big) = \frac{4 + 2.1\,(e-1)}{AB^m} + (e^{1/B}-1) < \frac{7.61}{AB^m} + (e^{1/B}-1).$$

For example, for $B = 2$, the last term $\sqrt{e} - 1 < 0.65$. Hence, in case $m \ge 3$, we need $\frac{7.61}{8A} \le 0.35$, where $A = 2.72$ fits well. Then we obtain (6.4) for $r = m$, i.e.,

$$|(\log f(t))^{(m)} - i^m\gamma_m| \le A\cdot 2^m(m-1)!\,\beta_{m+\alpha}|t|^\alpha \tag{6.7}$$

for all $m \ge 1$ and with any $A \ge 2.72$, once we have this inequality for the first two values $m = 1$ and $m = 2$ (induction hypothesis).

When $m = 1$, according to (6.1) with $s = 1 + \alpha$, we have $|f'(t)| \le 2\beta_{1+\alpha}|t|^\alpha$, so

$$|(\log f(t))'| = \frac{|f'(t)|}{|f(t)|} \le \frac{2\beta_{1+\alpha}|t|^\alpha}{1 - \frac{\varepsilon^2}{2}} \le 2.05\,\beta_{1+\alpha}|t|^\alpha,$$

so (6.7) is fulfilled. When $m = 2$,

$$(\log f(t))'' + \sigma^2 = \frac{f''(t)f(t) - f'(t)^2}{f(t)^2} + \sigma^2$$
$$= \frac{(f''(t) + \sigma^2)f(t) + \sigma^2 f(t)(f(t) - 1) - f'(t)^2}{f(t)^2}.$$

According to (6.1), $|f''(t) + \sigma^2| \leq 2\beta_{2+\alpha}|t|^\alpha$ and $|f(t) - 1| \leq 2\beta_\alpha|t|^\alpha$. Hence,

$$|(\log f(t))'' + \sigma^2| \leq \frac{2\beta_{2+\alpha}|t|^\alpha + 2\sigma^2\beta_\alpha|t|^\alpha + 2\beta_1\beta_{1+\alpha}|t|^\alpha}{|f(t)|^2}$$
$$\leq \frac{6\beta_{2+\alpha}|t|^\alpha}{(1 - \frac{\varepsilon^2}{2})^2} \leq 6.25\, \beta_{2+\alpha}|t|^\alpha.$$

In both cases, (6.7) is fulfilled with $A = 2.72$. Thus, we have proved:

Lemma 6.1 *Let X be a random variable such that $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2$ ($\sigma > 0$), and $\beta_{m+\alpha} < \infty$ for some integer $m \geq 2$ and $0 < \alpha \leq 1$. Then, in the interval $\sigma|t| \leq \frac{1}{5}$, the characteristic function $f(t)$ of X is not vanishing and satisfies*

$$\left|\frac{d^m}{dt^m} \log f(t) - i^m\gamma_m\right| \leq 2.72 \cdot 2^m\,(m-1)!\,\beta_{m+\alpha}|t|^\alpha.$$

This inequality remains to hold for $m = 1$ as well, if $\mathbb{E}X^2$ is finite.

Now, if s is integer, for any $p = 0, 1, \ldots, s$, the function

$$w(t) = \frac{d^p}{dt^p} \log f(t) - \frac{d^p}{dt^p} \sum_{k=2}^{s-1} \gamma_k \frac{(it)^k}{k!}$$

has zero derivatives at $t = 0$ up to order $s - p - 1$, while $w^{(s-p)}(t) = \frac{d^s}{dt^s} \log f(t)$. Hence, by Proposition 4.1 and Taylor's formula,

$$|w(t)| \leq \sup_{|z|\leq|t|} |w^{(s-p)}(z)|\, \frac{|t|^{s-p}}{(s-p)!} \leq (s-1)!\,\beta_s\, \frac{|t|^{s-p}}{(s-p)!}, \qquad \text{if } \sigma|t| \leq \frac{1}{5}.$$

In the general case $s = m + \alpha$ with integer $m \geq 2$ and $0 < \alpha \leq 1$, for any $p = 0, 1, \ldots, m$, consider the function

$$w(t) = \frac{d^p}{dt^p} \log f(t) - \frac{d^p}{dt^p} \sum_{k=2}^{m} \gamma_k \frac{(it)^k}{k!}.$$

It has zero derivatives at $t = 0$ up to order $m-p-1$, while $w^{(m-p)}(t) = \frac{d^m}{dt^m}\log f(t) - \gamma_m i^m$. Hence, for $p \le m-1$, by Taylor's integral formula,

$$\begin{aligned} w(t) &= \frac{t^{m-p}}{(m-p-1)!}\int_0^1 (1-u)^{m-p-1}\, w^{(m-p)}(tu)\,du \\ &= \frac{t^{m-p}}{(m-p-1)!}\int_0^1 (1-u)^{m-p-1}\,\big((\log f)^{(m)}(tu) - \gamma_m i^m\big)\,du. \end{aligned}$$

Applying Lemma 6.1, we then get that

$$\begin{aligned} |w(t)| &\le \frac{|t|^{m-p}}{(m-p-1)!}\int_0^1 (1-u)^{m-p-1}\, 2.72\cdot 2^m (m-1)!\,\beta_s\,|tu|^\alpha\,du \\ &= 2.72\cdot 2^m\,(m-1)!\,\beta_s\,|t|^{s-p}\,\frac{\Gamma(\alpha+1)}{\Gamma(s-p+1)}. \end{aligned}$$

The obtained inequality is also true for $p = m$ (Lemma 6.1). Using $\Gamma(\alpha+1) \le 1$, we arrive at:

Proposition 6.2 *Let f be the characteristic function of a random variable X with $\mathbb{E}X = 0$ and $\beta_s = \mathbb{E}\,|X|^s < \infty$ for some $s > 2$. Put $s = m + \alpha$ with m integer and $0 < \alpha \le 1$. Then in the interval $\sigma|t| \le \frac{1}{5}$,*

$$\log f(t) = \sum_{k=2}^m \gamma_k\,\frac{(it)^k}{k!} + \varepsilon(t)$$

with

$$\Big|\frac{d^p}{dt^p}\,\varepsilon(t)\Big| \le 2.72\cdot 2^m (m-1)!\,\beta_s\,\frac{|t|^{s-p}}{\Gamma(s-p+1)}$$

for all $p = 0, 1, \dots, m$. If $\alpha = 1$, in the same interval, for all $p = 0, 1, \dots, m+1$,

$$\Big|\frac{d^p}{dt^p}\,\varepsilon(t)\Big| \le m!\,\beta_s\,\frac{|t|^{s-p}}{\Gamma(s-p+1)}.$$

Let us state particular cases in this statement corresponding to the values $s = 3$ and $s = 4$.

Corollary 6.3 *Let $f(t)$ be the characteristic function of a random variable X with $\mathbb{E}X = 0$. If $\beta_3 = \mathbb{E}\,|X|^3 < \infty$, then in the interval $\sigma|t| \le \frac{1}{5}$,*

$$\log f(t) = -\frac{\sigma^2 t^2}{2} + \varepsilon(t) \quad \text{with} \quad \Big|\frac{d^p}{dt^p}\,\varepsilon(t)\Big| \le 6\beta_3\,\frac{|t|^{3-p}}{(3-p)!}, \qquad p = 0,1,2,3.$$

Moreover, if $\beta_4 = \mathbb{E}X^4 < \infty$*, then*

$$\log f(t) = -\frac{\sigma^2 t^2}{2} + \alpha_3 \frac{(it)^3}{6} + \varepsilon(t) \quad \textit{with} \ \left|\frac{d^p}{dt^p}\,\varepsilon(t)\right| \leq 24\,\beta_4\,\frac{|t|^{4-p}}{(4-p)!},$$
$$p = 0, 1, 2, 3, 4.$$

PART II. Lyapunov coefficients and products of characteristic functions

7 Properties of Lyapunov Coefficients

From now on, we deal with a sequence $X_1, \dots, X_n$ of independent random variables such that $\mathbb{E}X_k = 0$, $\mathbb{E}X_k^2 = \sigma_k^2$ ($\sigma_k \geq 0$) and $\sum_{k=1}^n \sigma_k^2 = 1$. The latter insures that the sum

$$S_n = X_1 + \cdots + X_n$$

has the first two moments $\mathbb{E}S_n = 0$ and $\mathbb{E}S_n^2 = 1$. For $s \geq 2$, consider the absolute moments $\beta_{s,k} = \mathbb{E}\,|X_k|^s$ and the corresponding Lyapunov coefficients

$$L_s = \sum_{k=1}^n \mathbb{E}\,|X_k|^s.$$

First, below we state a few simple, but useful auxiliary results about these quantities.

Proposition 7.1 *The function* $L_s^{\frac{1}{s-2}}$ *is non-decreasing in* $s > 2$*. In particular,* $L_3 \leq L_s^{\frac{1}{s-2}}$ *for all* $s \geq 3$.

Proof Let F_k denote the distribution of X_k. By the basic assumption on the variances σ_k^2, the equality $d\mu(x) = \sum_{k=1}^n x^2\,dF_k(x)$ defines a probability measure on the real line. Moreover,

$$L_s = \sum_{k=1}^n \int_{-\infty}^{\infty} |x|^s\,dF_k(x) = \int_{-\infty}^{\infty} |x|^{s-2}\,d\mu(x) = \mathbb{E}\,|\xi|^{s-2},$$

where ξ is a random variable distributed according to μ. Hence, $L_s^{\frac{1}{s-2}} = (\mathbb{E}\,|\xi|^{s-2})^{\frac{1}{s-2}}$. Here the right-hand side represents a non-decreasing function in s. □

Proposition 7.2 *We have* $\max_k \sigma_k \leq L_s^{1/s}$ $(s \geq 2)$*. In particular,* $L_3^{1/3} \geq \max_k \sigma_k$.

Proof Using $\sigma_k^s \leq \beta_{s,k}$, we have $\max_k \sigma_k \leq \big(\sum_{k=1}^n \sigma_k^s\big)^{1/s} \leq \big(\sum_{k=1} \beta_{s,k}\big)^{1/s} = L_s^{1/s}$. □

There is also a uniform lower bound on the Lyapunov coefficients depending upon n, only.

Proposition 7.3 *We have $L_s \geq n^{-\frac{s-2}{2}}$ $(s \geq 2)$. In particular, $L_3 \geq \frac{1}{\sqrt{n}}$ and $L_4 \geq \frac{1}{n}$.*

Proof Let $s > 2$. By Hölder's inequality with exponents $p = \frac{s}{s-2}$ and $q = \frac{s}{2}$,

$$1 = \sum_{k=1}^{n} \sigma_k^2 \leq n^{1/p} \bigg(\sum_{k=1}^{n} \sigma_k^s \bigg)^{1/q} \leq n^{1/p} \bigg(\sum_{k=1}^{n} \beta_{s,k} \bigg)^{1/q} = n^{1/p} L_s^{1/q}.$$

Hence, $L_s \geq n^{-q/p}$. □

Note that the finiteness of the moments $\beta_{s,k}$ for all $k \leq n$ is equivalent to the finiteness of the Lyapunov coefficient L_s. In this case, one may introduce the corresponding cumulants

$$\gamma_{p,k} = \gamma_p(X_k) = \frac{d^p}{i^p\, dt^p} \log v_k(t)\Big|_{t=0}, \qquad p = 0, 1, 2, \dots, [s],$$

where $v_k = \mathbb{E}\, e^{itX_k}$ denote the characteristic functions of X_k. Since the characteristic function of S_n is given by the product

$$f_n(t) = \mathbb{E}\, e^{itS_n} = v_1(t) \dots v_n(t),$$

the cumulants of S_n exist for the same values of p and are given by

$$\gamma_p = \gamma_p(S_n) = \frac{d^p}{i^p\, dt^p} \log f_n(t)\Big|_{t=0} = \sum_{k=1}^{n} \gamma_{p,k}.$$

The first values are $\gamma_0 = \gamma_1 = 0$, $\gamma_2 = 1$.

Applying Proposition 3.2 (Bikjalis inequality), we immediately obtain a similar relation between the Lyapunov coefficients and the cumulants of the sums.

Proposition 7.4 *For all $p = 2, \dots, [s]$,*

$$|\gamma_p| \leq (p-1)!\, L_p. \tag{7.1}$$

The Lyapunov coefficients may also be used to bound absolute moments of the sums S_n. In particular, there is the following observation due to Rosenthal [R].

Proposition 7.5 *With some constants A_s depending on s, only,*

$$\mathbb{E}\, |S_n|^s \leq A_s \max\{L_s, 1\}. \tag{7.2}$$

Moment inequalities of the form (7.2) are called Rosenthal's or Rosenthal-type inequalities. The study of the best value A_s has a long story, and here we only mention several results.

Define A_s^* to be an optimal constant in (7.2), when it is additionally assumed that the distributions of X_k are symmetric about the origin. By Jensen's inequality, for the optimal constant A_s there is a simple general relation

$$A_s^* \le A_s \le 2^{s-1} A_s^*,$$

which reduces in essence the study of Rosenthal-type inequalities to the symmetric case.

Johnson, Schechtman, and Zinn [J-S-Z] have derived the two-sided bounds

$$\frac{s}{\sqrt{2}\,e\,\log(\max(s,e))} \le (A_s^*)^{1/s} \le \frac{7.35\,s}{\log(\max(s,e))}.$$

Hence, asymptotically $A_s^{1/s}$ is of order $s/\log s$ for growing values of s. They have also obtained an upper bound with a better numerical factor, $(A_s^*)^{1/s} \le s/\sqrt{\log\max(s,e)}$, which implies a simple bound

$$A_s \le (2s)^s, \quad s > 2. \tag{7.3}$$

As for the best constant in the symmetric case, it was shown by Ibragimov and Sharakhmetov [I-S] that $A_s^* = \mathbb{E}\,|\xi - \eta|^s$ for $s > 4$, where ξ and η are independent Poisson random variables with parameter $\lambda = \frac{1}{2}$ (cf. also [Pi] for a similar description without the symmetry assumption). In particular, $(A_s^*)^{1/s} \sim \frac{s}{e\log s}$ as s tends to infinity. This result easily yields

$$A_s^* \le s! \qquad \text{for } s = 3,4,5,\dots,$$

and thus $A_s \le 2^{s-1} s!$ For even integers s, there is an alternative argument. Applying the expression (3.2) to S_n (for the cumulants in terms of the moments) and recalling (7.1), we get

$$\mathbb{E}\,|S_n|^s = \alpha_s(S_n) = s! \sum \prod_{r=1}^{s} \frac{1}{k_r!}\Big(\frac{\gamma_r(S_n)}{r!}\Big)^{k_r} \le s! \sum \prod_{r=1}^{s} \frac{1}{k_r!}\Big(\frac{L_{r^*}}{r}\Big)^{k_r}, \tag{7.4}$$

where $r^* = \max(r,2)$, and where the summation is performed over all tuples $(k_1,\dots,k_s)$ of non-negative integers such that $k_1 + 2k_2 + \dots + sk_s = s$. (The left representation was emphasized in [P-U].) Now, by Proposition 7.1, $L_r \le L_s^{\frac{r-2}{s-2}} \le (\max(L_s,1))^{r/s}$. Hence, by Lemma 1.2 (cf. (1.5)), the last sum in (7.4) does not exceed

$$\sum \prod_{r=2}^{s} \frac{1}{k_r!} \left(\frac{(\max(L_s^{1/s}, 1))^r}{r} \right)^{k_r} \le \max(L_s, 1).$$

Hence, $A_s \le s!$ for $s = 4, 6, 8, \dots$

To involve real values of s, for our further purposes it will be sufficient to use the upper bound (7.3).

8 Logarithm of the Product of Characteristic Functions

We keep the same notations and assumptions as in the previous section. Let us return to the characteristic function

$$f_n(t) = \mathbb{E}\, e^{itS_n} = v_1(t) \dots v_n(t)$$

of the sum $S_n = X_1 + \dots + X_n$ in terms of the characteristic functions $v_k = \mathbb{E}\, e^{itX_k}$. To get the Taylor expansion for f_n, recall that, by Proposition 6.2, applied to each X_k, we have

$$v_k(t) = \exp\left\{ \sum_{l=2}^{m} \gamma_{l,k} \frac{(it)^l}{l!} + \varepsilon_k(t) \right\}. \tag{8.1}$$

As we know, the function ε_k has $[s]$ continuous derivative, satisfying in the interval $\sigma_k|t| \le \frac{1}{5}$

$$\left| \frac{d^p}{dt^p} \varepsilon_k(t) \right| \le 2.72 \cdot 2^m (m-1)!\, \beta_{s,k} \frac{|t|^{s-p}}{\Gamma(s-p+1)}, \qquad p = 0, 1, \dots, m.$$

This assertion also extends to the case $p = m+1$, when $\alpha = 1$ (with better constants). Multiplying the expansions (8.1) and using $\gamma_2 = 1$, we arrive at a similar expansion for f.

Lemma 8.1 *Assume that $L_s < \infty$ for some $s = m + \alpha$ with $m \ge 2$ integer and $0 < \alpha \le 1$. Then, in the interval $\max_k \sigma_k |t| \le \frac{1}{5}$, we have*

$$e^{t^2/2} f_n(t) = \exp\left\{ Q_m(it) + \varepsilon(t) \right\}, \qquad Q_m(it) = \sum_{l=3}^{m} \gamma_l \frac{(it)^l}{l!}, \tag{8.2}$$

where the function ε has $[s]$ continuous derivatives, satisfying for all $p = 0, 1, \dots, m$,

$$\left| \frac{d^p}{dt^p} \varepsilon(t) \right| \le 2.72 \cdot 2^m (m-1)!\, L_s \frac{|t|^{s-p}}{\Gamma(s-p+1)}. \tag{8.3}$$

In addition, if $s = m+1 \geq 3$, then in the same interval, for all $p = 0, 1, \dots, m+1$,

$$\Big|\frac{d^p}{dt^p}\,\varepsilon(t)\Big| \;\leq\; m!\,L_s\,\frac{|t|^{s-p}}{\Gamma(s-p+1)}. \tag{8.4}$$

Both bounds hold in the interval $L_s^{\frac{1}{s}}|t| \;\leq\; \frac{1}{5}$, since $L_s^{\frac{1}{s}} \;\geq\; \max_k \sigma_k$ (Proposition 8.2). In case $s \geq 3$, these bounds hold in the interval $L_3|t|^3 \leq 1$.

As a next natural step, we want to replace the term $e^{\varepsilon(t)}$ in (8.2) with a simpler one, $1+\varepsilon(t)$, keeping similar bounds on the remainder term as in (8.3)–(8.4). To this aim, in the smaller interval $L_s^{1/s}|t| \leq \frac{1}{8}$, we consider the function

$$\delta(t) = e^{\varepsilon(t)} - 1.$$

By Proposition 1.1, for any $p = 1, \dots, m$,

$$\delta^{(p)}(t) = \frac{d^p}{dt^p}\,e^{\varepsilon(t)} = p!\,e^{\varepsilon(t)}\sum\prod_{r=1}^{p}\frac{1}{k_r!}\Big(\frac{1}{r!}\,\varepsilon^{(r)}(t)\Big)^{k_r}, \tag{8.5}$$

where the summation is performed over all non-negative integer solutions $k = (k_1, \dots, k_p)$ to $k_1 + 2k_2 + \dots + pk_p = p$. By (8.3) with $p = 0$,

$$|\varepsilon(t)| \leq 2.72\cdot\frac{2^m}{m}\,L_s|t|^s \leq 2.72\,\frac{2^m}{m\,8^s} \leq 1.36\,\Big(\frac{1}{4}\Big)^s < 0.09,$$

since $s \geq m \geq 2$. Hence,

$$|\delta(t)| \;\leq\; e^{0.09}\,|\varepsilon(t)| \;\leq\; \frac{3\cdot 2^m}{m}\,L_s|t|^s.$$

As for derivatives of order $1 \leq r \leq m$, applying (8.3) and the bound $C_m^r \leq 2^{m-1}$, we have

$$\begin{aligned}\frac{1}{r!}\,|\varepsilon^{(r)}(t)| &\leq 2.72\cdot 2^m(m-1)!\,\frac{L_s|t|^{s-r}}{r!\,\Gamma(s-r+1)} \\ &\leq 2.72\cdot\frac{2^m}{m}\,\frac{m!}{r!\,(m-r)!}\,L_s|t|^{s-r} \;\leq\; 1.36\cdot\frac{4^m}{m}\,L_s|t|^{s-r}.\end{aligned}$$

Here $\lambda \equiv 1.36\cdot\frac{4^m}{m}\,L_s|t|^s \leq 0.68\cdot 2^{-m} \leq 0.68\cdot 2^{-p}$ whenever $1 \leq p \leq m$. Hence, by Lemma 1.2 with this value of λ and with $\lambda_0 = 0.68$ (cf. (1.4)), we have

$$\sum\prod_{r=1}^{p}\frac{1}{k_r!}\Big(1.36\cdot\frac{4^m}{m}\,L_s|t|^{s-r}\Big)^{k_r} \leq e^{0.17}\,1.36\cdot\frac{4^m}{m}\,L_s|t|^{s-p}.$$

As a result, from (8.5) we get

$$\frac{1}{p!}\,|\delta^{(p)}(t)| \le e^{|\varepsilon(t)|}\sum\prod_{r=1}^{p}\frac{1}{k_r!}\,\Big|\frac{1}{r!}\,\varepsilon^{(r)}(t)\Big|^{k_r}$$

$$\le e^{0.09}\sum\prod_{r=1}^{p}\frac{1}{k_r!}\Big(1.36\cdot\frac{4^m}{m}\,L_s|t|^{s-r}\Big)^{k_r}$$

$$= e^{0.09}\,e^{0.17}\,1.36\cdot\frac{4^m}{m}\,L_s|t|^{s-p} \le 2\cdot\frac{4^m}{m}\,L_s|t|^{s-p}.$$

As we have seen, the resulting bound also holds for $p = 0$ (with a better constant). More precisely, we thus get

$$\frac{1}{p!}\,|\delta^{(p)}(t)| \le 2\cdot\frac{4^m}{m}L_s|t|^{s-p} \quad (1 \le p \le m), \qquad |\delta(t)| \le 3\cdot\frac{2^m}{m}L_s|t|^s \quad (p = 0).$$

Scenario 2. In case $s = m + 1$ is integer, $m \ge 2$, one may involve an additional value $p = m + 1$. In case $p = 0$, (8.4) gives $|\varepsilon(t)| \le L_s|t|^s \le (\frac{1}{8})^3$, and then

$$|\delta(t)| \le e^{1/8^3}\,|\varepsilon(t)| \le 1.002\,L_s|t|^s.$$

For the derivatives of order $1 \le r \le m + 1$, we have

$$\frac{1}{r!}\,|\varepsilon^{(r)}(t)| \le m!\,\frac{L_s|t|^{s-r}}{r!\,\Gamma(s-r+1)}$$

$$= \frac{m!}{r!\,((m+1)-r)!}\,L_s|t|^{s-r} \le \frac{2^m}{m+1}L_s|t|^{s-r}.$$

Here $\frac{2^m}{m+1}L_s|t|^s \le \frac{1}{3}\,(\frac{2}{8})^m < \frac{1}{12}\,2^{-p}$, if $1 \le p \le m + 1$. Hence, by Lemma 1.2 with $\lambda_0 = \frac{1}{12}$,

$$\sum\prod_{r=1}^{p}\frac{1}{k_r!}\Big(\frac{2^m}{m+1}L_s|t|^{s-r}\Big)^{k_r} \le e^{1/48}\cdot\frac{2^m}{m+1}L_s|t|^{s-p}.$$

As a result, for any $p = 1,\dots,m + 1$,

$$\frac{1}{p!}\,|\delta^{(p)}(t)| \le e^{|\varepsilon(t)|}\sum\prod_{r=1}^{p}\frac{1}{k_r!}\Big|\frac{1}{r!}\,\varepsilon^{(r)}(t)\Big|^{k_r}$$

$$\le 1.002\sum\prod_{r=1}^{p}\frac{1}{k_r!}\Big(\frac{2^m}{m+1}L_s|t|^{s-r}\Big)^{k_r}$$

$$= 1.002\, e^{1/48} \cdot \frac{2^m}{m+1} L_s |t|^{s-p} \le 1.1 \cdot \frac{2^m}{m+1} L_s |t|^{s-p}.$$

We thus get

$$\frac{1}{p!}\, |\delta^{(p)}(t)| \le 1.1 \cdot \frac{2^m}{m+1} L_s |t|^{s-p} \quad (1 \le p \le m+1), \qquad |\delta(t)| \le 1.1\, L_s |t|^s \quad (p = 0).$$

Let us summarize, replacing δ with ε (as the notation, only).

Proposition 8.2 *Assume that $L_s < \infty$ for $s = m + \alpha$ with $m \ge 2$ integer and $0 < \alpha \le 1$. Then in the interval $L_s^{\frac{1}{s}} |t| \le \frac{1}{8}$, we have*

$$e^{t^2/2} f_n(t) = e^{Q_m(it)}\, (1 + \varepsilon(t)), \qquad Q_m(it) = \sum_{l=3}^{m} \gamma_l \, \frac{(it)^l}{l!}, \tag{8.6}$$

where the function ε has $[s]$ continuous derivatives, satisfying

$$\frac{1}{p!} \left| \frac{d^p}{dt^p}\, \varepsilon(t) \right| \le C_m L_s |t|^{s-p}, \qquad p = 0, 1, \dots, m,$$

with $C_m = 2 \cdot \frac{4^m}{m}$. Moreover, if $s = m + 1$, one may take $C_m = 1.1 \cdot \frac{2^m}{m+1}$ for all $0 \le p \le m + 1$. If $p = 0$, this bound holds with $C_m = 3 \cdot \frac{2^m}{m}$. Moreover, one may take $C_m = 1.1$ when $s = m + 1$.

9 The Case $2 < s \le 3$

For the values $2 < s \le 3$, the cumulant sum in (8.2) and (8.6) does not contain any term, that is, $Q_m = 0$, so

$$f_n(t) = e^{-t^2/2}\, (1 + \varepsilon(t)).$$

Let us specify Proposition 8.2 in this case. If $L_s < \infty$ for $s = 2 + \alpha$, $0 < \alpha \le 1$, we obtain that in the interval $L_s^{1/s} |t| \le \frac{1}{8}$, the function $\varepsilon(t)$ has $[s]$ continuous derivatives satisfying

$$|\varepsilon(t)| \le 6 L_s |t|^s, \qquad \left| \frac{d^p}{dt^p}\, \varepsilon(t) \right| \le 16\, L_s |t|^{s-p} \quad (p = 1, 2).$$

Moreover, in case $s = 3$,

$$|\varepsilon(t)| \le 1.1\, L_s |t|^3, \qquad \left| \frac{d^p}{dt^p}\, \varepsilon(t) \right| \le 1.5\, L_3 |t|^{3-p} \quad (p = 1, 2, 3).$$

Using these representations, one may easily derive the following two propositions.

Proposition 9.1 *Let $L_s < \infty$ for $s = 2 + \alpha$ $(0 < \alpha < 1)$. Then in the interval $L_s^{\frac{1}{s}}|t| \leq \frac{1}{8}$,*

$$\big|f_n(t) - e^{-t^2/2}\big| \leq 6L_s|t|^s e^{-t^2/2},$$

$$\Big|\frac{d}{dt}\big(f_n(t) - e^{-t^2/2}\big)\Big| \leq 16\,L_s\big(|t|^{s-1} + |t|^{s+1}\big)\,e^{-t^2/2},$$

$$\Big|\frac{d^2}{dt^2}\big(f_n(t) - e^{-t^2/2}\big)\Big| \leq 32\,L_s\big(|t|^{s-2} + |t|^{s+2}\big)\,e^{-t^2/2}.$$

Proof Introduce the function $h(t) = f_n(t) - e^{-t^2/2} = e^{-t^2/2}\,\varepsilon(t)$. The first inequality is immediate. Next,

$$e^{t^2/2}\,|h'(t)| = |\varepsilon'(t) - t\varepsilon(t)| \leq 16L_s\,(|t|^{s-1} + |t|^{s+1}).$$

For the second derivative, we get

$$\begin{aligned} e^{t^2/2}\,|h''(t)| &\leq |\varepsilon''(t)| + 2|t|\,|\varepsilon'(t)| + |t^2 - 1|\,|\varepsilon(t)| \\ &\leq 16L_s\big(|t|^{s-2} + 2|t|\,|t|^{s-1} + |t^2 - 1|\,|t|^s\big) \\ &= 16L_s|t|^{s-2}\big(1 + 2t^2 + |t^2 - 1|\,t^2\big). \end{aligned}$$

If $|t| \leq 1$, then the expression in the last brackets is equal to $1 + 2t^2 - t^4 \leq 2\,(1 + t^4)$. If $|t| \geq 1$, it is equal to $1 + t^2 + t^4 \leq 2(1 + t^4)$. □

Proposition 9.2 *Let $L_3 < \infty$. Then in the interval $L_3^{1/3}|t| \leq \frac{1}{8}$,*

$$\big|f_n(t) - e^{-t^2/2}\big| \leq 1.1\,L_3|t|^3\,e^{-t^2/2},$$

$$\Big|\frac{d}{dt}\big(f_n(t) - e^{-t^2/2}\big)\Big| \leq 1.5\,L_3\,(t^2 + t^4)\,e^{-t^2/2},$$

$$\Big|\frac{d^2}{dt^2}\big(f_n(t) - e^{-t^2/2}\big)\Big| \leq 3L_3\,(|t| + |t|^5)\,e^{-t^2/2},$$

$$\Big|\frac{d^3}{dt^3}\big(f_n(t) - e^{-t^2/2}\big)\Big| \leq 12L_3\,(1 + t^6)\,e^{-t^2/2}.$$

Proof Again, consider the function $h(t) = f_n(t) - e^{-t^2/2} = e^{-t^2/2}\,\varepsilon(t)$. The case $p = 0$ is immediate. For $p = 1$,we have

$$e^{t^2/2}\,|h'(t)| = |\varepsilon'(t) - t\varepsilon(t)| \leq 1.5\,L_3\,(t^2 + t^4).$$

For $p = 2$, we get, using the previous arguments,

$$e^{t^2/2}\,|h''(t)| \le |\varepsilon''(t)| + 2|t|\,|\varepsilon'(t)| + |t^2-1|\,|\varepsilon(t)|$$
$$\le 1.5\,L_3\,\big(|t| + 2|t|\,t^2 + |t^2-1|\,|t|^3\big) \le 3L_3|t|\,\big(1+t^4\big).$$

Finally, for $p = 3$, using $|\varepsilon^{(p)}(t)| \le 2.2\,L_3|t|^{3-p}$ for $p = 0, 1, 2, 3$, we get

$$e^{t^2/2}\,|h'''(t)| \le |\varepsilon'''(t)| + 3|t|\,|\varepsilon''(t)| + 3\,|t^2-1|\,|\varepsilon'(t)| + |t^3-3t|\,|\varepsilon(t)|$$
$$\le 1.5\,L_3\,\Big(1 + 3t^2 + 3\,|t^2-1|\,t^2 + |t^3-3t|\,|t|^3\Big).$$

If $|t| \le 1$, the expression in the brackets equals and does not exceed $1 + 6t^2 - t^6 \le 1 + 4\sqrt{2} < 8$. If $|t| \ge 1$, it does not exceed $1 + 6t^4 + t^6 \le 8t^6$. □

PART III. "Corrected normal characteristic" functions

10 Polynomials P_m in the Normal Approximation

Let us return to the approximation given in Proposition 8.2, i.e.,

$$e^{t^2/2} f_n(t) = e^{Q_m(it)}\,(1+\varepsilon(t)), \quad \text{where} \quad Q_m(it) = \sum_{l=3}^{m} \gamma_l\,\frac{(it)^l}{l!} \quad (\gamma_l = \gamma_l(S_n)).$$

We are now going to simplify the expression $e^{Q_m(it)}\,(1+\varepsilon(t))$ to the form $1+P_m(it)+\varepsilon(t)$ with a certain polynomial P_m and with a new remainder term, which would be still as small as the Lyapunov coefficient L_s (including the case of derivatives). This may indeed be possible on a smaller interval in comparison with $L_s^{1/s}|t| \le 1$. In view of Propositions 9.1–9.2, one may naturally assume that $s > 3$, so that $s = m + \alpha$, $m \ge 3$ (integer), $0 < \alpha \le 1$.

Using Taylor's expansion for the exponential function, one can write

$$e^{Q_m(it)} = \sum_{k_1=0}^{\infty} \Big(\frac{\gamma_3}{3!}\Big)^{k_1} \frac{(it)^{3k_1}}{k_1!} \cdots \sum_{k_{s-3}=0}^{\infty} \Big(\frac{\gamma_m}{m!}\Big)^{k_{m-2}} \frac{(it)^{mk_{m-2}}}{k_{m-2}!}$$
$$= \sum_{k_1,\dots,k_{m-2}\ge 0} \frac{\gamma_3^{k_1}\dots\gamma_m^{k_{m-2}}}{3!^{k_1}\dots m!^{k_{m-2}}}\,\frac{(it)^{3k_1+\dots+mk_{m-2}}}{k_1!\dots k_{m-2}!} = \sum_{k=0}^{\infty} a_k\,(it)^k$$

with coefficients

$$a_k = \sum_{3k_1+\dots+mk_{m-2}=k} \frac{1}{k_1!\dots k_{m-2}!}\Big(\frac{\gamma_3}{3!}\Big)^{k_1}\dots\Big(\frac{\gamma_m}{m!}\Big)^{k_{m-2}}.$$

Clearly, all these series are absolutely convergent for all t. A certain part of the last infinite series represents the desired polynomial P_m.

Definition 10.1 Put

$$P_m(it) = \sum \frac{1}{k_1! \dots k_{m-2}!} \left(\frac{\gamma_3}{3!}\right)^{k_1} \dots \left(\frac{\gamma_m}{m!}\right)^{k_{m-2}} (it)^{3k_1+\dots+mk_{m-2}},$$

where the summation runs over all collections of non-negative integers $(k_1, \dots, k_{m-2})$ that are not all zero and such that $d \equiv k_1 + 2k_2 + \dots + (m-2)k_{m-2} \le m-2$.

Here the constraint $d \le m-2$ has the aim to involve only those terms and coefficients in P_m that may not be small in comparison with L_s. Indeed, as we know from Proposition 7.4,

$$|\gamma_l| \le (l-1)!\, L_l \le (l-1)!\, L_s^{(l-2)/(s-2)}, \qquad 3 \le l \le [s],$$

which gives

$$\left|\left(\frac{\gamma_3}{3!}\right)^{k_1} \dots \left(\frac{\gamma_m}{m!}\right)^{k_{m-2}}\right| \le \frac{L_s^{d/(s-2)}}{3^{k_1} \dots m^{k_{m-2}}}. \tag{10.1}$$

So, the left product is at least as small as L_s in case $d \ge m-1$, when L_s is small. Of course, this should be justified when comparing $e^{Q_m(it)}$ and $1 + P_m(it)$ on a proper interval of the t-axis. This will be done in the next two sections.

The index m for P indicates that all cumulants up to γ_m participate in the constructions of these polynomials. The power

$$k = 3k_1 + \dots + mk_{m-2} = d + 2(k_1 + k_2 + \dots + k_{m-2})$$

may vary from 3 to $3(m-2)$, with maximum $3(m-2)$ attainable when $k_1 = m-2$ and all other $k_r = 0$. Anyway, $\deg(P_m) \le 3(m-2)$.

These observations imply a simple general bound on the growth of P_m, which will be needed in the sequel. First, $|t|^k \le \max\left\{|t|^3, |t|^{3(m-2)}\right\}$. Hence, by Definition 10.1,

$$|P_m(it)| \le \max\left\{|t|^3, |t|^{3(m-2)}\right\} \sum \frac{1}{k_1! \dots k_{m-2}!} \frac{L_s^{d/(s-2)}}{3^{k_1} \dots m^{k_{m-2}}}.$$

Using the elementary bound

$$\sum \frac{1}{k_1! \dots k_{m-2}!} \frac{1}{3^{k_1} \dots m^{k_{m-2}}} < e^{1/3} \dots e^{1/m} < m, \tag{10.2}$$

we arrive at:

Proposition 10.2 *For all t real,*

$$|P_m(it)| \leq m \max\left\{|t|^3, |t|^{3(m-2)}\right\} \max\left\{L_s^{\frac{1}{s-2}}, L_s^{\frac{m-2}{s-2}}\right\}.$$

Let us describe the first three polynomials. Clearly, $P_3(it) = \gamma_3 \frac{(it)^3}{3!}$, while for $m = 4$,

$$P_4(it) = \sum_{0<k_1+2k_2\leq 2} \frac{1}{k_1!\,k_2!} \left(\frac{\gamma_3}{3!}\right)^{k_1} \left(\frac{\gamma_4}{4!}\right)^{k_2} (it)^{3k_1+4k_2} = \gamma_3 \frac{(it)^3}{3!} + \gamma_4 \frac{(it)^4}{4!} + \gamma_3^2 \frac{(it)^6}{2!\,3!^2}.$$

Correspondingly, for $m = 5$,

$$\begin{aligned} P_5(it) &= \sum_{0<k_1+2k_2+3k_3\leq 3} \frac{1}{k_1!\,k_2!\,k_3!} \left(\frac{\gamma_3}{3!}\right)^{k_1} \left(\frac{\gamma_4}{4!}\right)^{k_2} \left(\frac{\gamma_5}{5!}\right)^{k_3} (it)^{3k_1+4k_2+5k_3} \\ &= \gamma_3 \frac{(it)^3}{3!} + \gamma_4 \frac{(it)^4}{4!} + \gamma_5 \frac{(it)^5}{5!} + \gamma_3^2 \frac{(it)^6}{2!\,3!^2} + \gamma_3^3 \frac{(it)^9}{3!\,3!^3}. \end{aligned}$$

11 Cumulant Polynomials Q_m

Properties of the polynomials P_m will be explored via the study of the cumulant polynomials

$$Q_m(z) = \sum_{l=3}^{m} \frac{\gamma_l}{l!} z^l,$$

which will be treated as polynomials in the complex variable z. In this section we collect auxiliary facts, assuming that $L_s < \infty$ for some $s = m + \alpha$, $m \geq 3$, where m is integer and $0 < \alpha \leq 1$. In that case, the first term in Q_m is $\frac{\gamma_3}{3!} z^3$.

Lemma 11.1 *If* $|z| \max\left\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\right\} \leq \frac{1}{4}$, *then* $|Q_m(z)| < 0.007$. *Moreover,*

$$|Q_m(z)| \leq 0.42\, L_s^{\frac{1}{s-2}} |z|^3.$$

Proof Since $s \to L_s^{\frac{1}{s-2}}$ is non-decreasing, we have $|z| \max\left\{L_m^{\frac{1}{m-2}}, L_m^{\frac{1}{3(m-2)}}\right\} \leq \frac{1}{4}$. As we know, for any integer $3 \leq l \leq m$,

$$|\gamma_l| \leq (l-1)!\, L_l \leq (l-1)!\, L_m^{\frac{l-2}{m-2}}.$$

Hence,

$$|Q_m(z)| \ \le \ \sum_{l=3}^m \frac{1}{l}\, L_l |z|^l \le \sum_{l=3}^m \frac{1}{l}\, L_m^{\frac{l-2}{m-2}} |z|^l = L_m^{\frac{1}{m-2}} |z|^3 \sum_{l=3}^m \frac{1}{l} \left(L_m^{\frac{1}{m-2}} |z|\right)^{l-3}$$
$$\le 0.42\, L_m^{\frac{1}{m-2}} |z|^3 \ \le \ 0.42\, L_s^{\frac{1}{s-2}} |z|^3,$$

where we used $L_m^{\frac{1}{m-2}} |z| \ \le\ \frac{1}{4}$ together with $\sum_{l=3}^\infty \frac{4^{-(l-3)}}{l} \ = \ 64 \log \frac{4}{3} - 18 \ < \ 0.42$. This gives the second assertion. Finally, apply $L_s^{\frac{1}{s-2}} |z|^3 \ \le\ \frac{1}{64}$ to get the uniform bound on $|Q_m(z)|$. □

Lemma 11.2 *In the interval* $|t| \max\big\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\big\} \le \frac{1}{8}$, *we have*

$$e^{Q_m(it)} \ = \ \sum_{k=0}^{m-2} \frac{Q_m(it)^k}{k!} \, + \varepsilon(t)$$

with

$$\frac{1}{p!} \left|\frac{d^p}{dt^p}\, \varepsilon(t)\right| \ \le \ 4^{s-2}\, L_s |t|^{3(s-2)-p}, \qquad p = 0, 1, \dots$$

Proof Consider the function of the complex variable $\Psi(w) = e^w - \sum_{k=0}^{m-2} \frac{w^k}{k!} = \sum_{k=m-1}^\infty \frac{w^k}{k!}$. If $|w| \le 1$, then $|w|^k \le |w|^{m-1} \le |w|^{s-2}$ for all $k \ge m-1$, so,

$$|\Psi(w)| \le |w|^{s-2} \sum_{k=m-1}^\infty \frac{1}{k!} \le |w|^{s-2}.$$

This inequality will be used with $w = Q_m(z)$. The function $\Psi(Q_m(z))$ is analytic in the complex plane. So, we may apply Cauchy's contour integral formula

$$\frac{d^p}{dt^p} \Psi(Q_m(it)) \ = \ \frac{p!}{2\pi} \int_{|z-it|=\rho} \frac{\Psi(Q_m(z))}{(z-it)^{p+1}}\, dz$$

with an arbitrary $\rho > 0$, which gives

$$\left|\frac{d^p}{dt^p} \Psi(Q_m(it))\right| \le \frac{p!}{\rho^p} \max_{|z-it|=\rho} |\Psi(Q_m(z))|.$$

Assume that $|t| > 0$ and choose $\rho = |t|$. Then on the circle $|z - it| = \rho$, necessarily $|z| \le 2|t|$ and, by the assumption on t,

$$|z| \max\big\{L_m^{\frac{1}{m-2}}, L_m^{\frac{1}{3(m-2)}}\big\} \le 2|t| \max\big\{L_m^{\frac{1}{m-2}}, L_m^{\frac{1}{3(m-2)}}\big\} \le \frac{1}{4}.$$

Hence, we may apply the uniform estimate of Lemma 11.1, $|Q_m(z)| \leq 0.007 < 1$, so that, involving also the non-uniform estimate of the same lemma, we get

$$\begin{aligned}|\Psi(Q_m(z))| &\leq |Q_m(z)|^{s-2} \leq \Big(0.42\, L_s^{\frac{1}{s-2}}\, |z|^3\Big)^{s-2} \\ &\leq (0.42)^{s-2} \cdot L_s \cdot (2|t|)^{3(s-2)} = 3.36^{s-2}\, L_s \cdot |t|^{3(s-2)}.\end{aligned}$$

As a result,

$$\left|\frac{d^p}{dt^p}\Psi(Q_m(it))\right| \leq \frac{p!}{|t|^p}\, 3.36^{s-2} L_s |t|^{3(s-2)}.$$

□

Note that, using $\sum_{k=m-1}^{\infty} \frac{1}{k!} \leq \frac{1.5}{(m-1)!}$, the assertion of Lemma 11.2 could be sharpened to

$$\frac{1}{p!}\left|\frac{d^p}{dt^p}\,\varepsilon(t)\right| \leq 3.2 \cdot \frac{4^s}{(m+1)!}\, L_s |t|^{3(s-2)-p}, \qquad p = 0, 1, \dots$$

Lemma 11.3 *In the interval* $|t| \max\big\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\big\} \leq \frac{1}{8}$, *we have*

$$\left|\frac{d^p}{dt^p}\, e^{Q_m(it)}\right| \leq 1.01\, p!\, |t|^{-p}, \qquad p = 1, 2, \dots$$

Proof By Cauchy's contour integral formula, for any $\rho > 0$,

$$\left|\frac{d^p}{dt^p}\, e^{Q_m(it)}\right| \leq \frac{p!}{\rho^r}\, \exp\Big\{\max_{|z-it|=\rho} |Q_m(z)|\Big\}.$$

Assume $|t| > 0$ and choose again $\rho = |t|$. Then on the cicrle $|z - it| = \rho$ we have

$$|z| \max\big\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\big\} \leq 2|t| \max\big\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\big\} \leq \frac{1}{4}.$$

Hence, we may apply the uniform estimate of Lemma 11.1 and notice that $e^{0.007} < 1.01$. □

12 Relations Between P_m and Q_m

The basic relation between polynomials P_m and Q_m is described in the following statement.

Proposition 12.1 *If* $L_s < \infty$ $(s > 3)$, *then for* $|t| \max\big\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\big\} \leq \frac{1}{8}$, *we have*

$$e^{Q_m(it)} = 1 + P_m(it) + \delta(t)$$

with

$$|\delta(t)| \ \leq \ 0.2 \cdot 4^s L_s \max\left\{|t|^s, |t|^{3(s-2)}\right\}.$$

Moreover, for all $p = 1, \dots, [s]$,

$$\frac{1}{p!}\left|\frac{d^p}{dt^p}\,\delta(t)\right| \ \leq \ 0.5 \cdot 7^s L_s \ \max\left\{|t|^{s-p}, |t|^{3(s-2)-p}\right\}.$$

Proof In view of Lemma 11.2, we may only be concerned with the remainder term

$$r(t) = \sum_{k=1}^{m-2} \frac{Q_m(it)^k}{k!} - P_m(it),$$

which we consider in the complex plane (by replacing it with $z \in \mathbb{C}$). Using the polynomial formula, let us represent the above sum as

$$\sum_{k=1}^{m-2} \frac{1}{k!}\left(\sum_{l=3}^{m} \gamma_l \, \frac{z^l}{l!}\right)^k = \sum_{k=1}^{m-2} \ \sum_{k_1+\dots+k_{m-2}=k} \frac{1}{k_1! \dots k_{m-2}!}\left(\frac{\gamma_3}{3!}\right)^{k_1} \dots \left(\frac{\gamma_m}{m!}\right)^{k_{m-2}} z^{3k_1+\dots+mk_{m-2}}.$$

Here the double sum almost defines $P_m(it)$ with the difference that Definition 10.1 contains the constraint $k_1 + 2k_2 + \dots + (m-2)k_{m-2} \leq m-2$, while now we have a weaker constraint $k_1 + k_2 + \dots + k_{m-2} \leq m-2$. Hence, all terms appearing in $P_m(it)$ are present in the above double sum, so

$$r(t) = \sum \frac{1}{k_1! \dots k_{m-2}!}\left(\frac{\gamma_3}{3!}\right)^{k_1} \dots \left(\frac{\gamma_m}{m!}\right)^{k_{m-2}} (it)^{3k_1+\dots+mk_{m-2}}$$

with summation subject to

$$k_1 + k_2 + \dots + k_{m-2} \leq m-2, \qquad k_1 + 2k_2 + \dots + (m-2)k_{m-2} \geq m-1.$$

Necessarily, all $k_j \leq m-2$ and at least one $k_j \geq 1$. Using $|\gamma_l| \leq (l-1)!\, L_s^{\frac{l-2}{s-2}}$, we get

$$|r(z)| \ \leq \ \sum \frac{1}{k_1! \dots k_{m-2}!} \prod_{l=3}^{m} L_s^{k_{l-2}\,\frac{l-2}{s-2}}\, |z|^N \ = \sum \frac{1}{k_1! \dots k_{m-2}!}\, L_s^M |z|^N,$$

where

$$M \ = \ M(k_1, \dots, k_{m-2}) \ = \ \frac{1}{s-2}\,(k_1 + 2k_2 + \dots + (m-2)k_{m-2}),$$

$$\begin{aligned} N \;=\; N(k_1,\dots,k_{m-2}) &= 3k_1+\dots+mk_{m-2} \\ &= (k_1+2k_2+\dots+(m-2)k_{m-2}) \\ &\quad +2\,(k_1+k_2+\dots+k_{m-2}). \end{aligned}$$

Note that $m+1 \le N \le m(m-2)$, which actually will not be used, and $(s-2)M = N-2k$. If $L_s^{\frac{1}{s-2}}|z| \le 1$, using the property $1 \le k \le s-2$, we have

$$L_s^{M-1}|z|^N \;\le\; |z|^{N-(s-2)(M-1)} \;=\; |z|^{(s-2)+2k} \;\le\; \max\left\{|z|^s, |z|^{3(s-2)}\right\}.$$

Hence

$$|r(z)| \;\le\; L_s \max\left\{|z|^s, |z|^{3(s-2)}\right\} \sum \prod_{l=3}^{m} \frac{1}{k_{l-2}!}\left(\frac{1}{l}\right)^{k_{l-2}}.$$

The latter sum is dominated by $e^{m-2} \le e^{s-2}$, so

$$|r(z)| \;\le\; e^{s-2} L_s \max\left\{|z|^s, |z|^{3(s-2)}\right\},$$

which can be used to prove Proposition 12.1 in case $p = 0$. Indeed, by Lemma 11.2 with its function $\varepsilon(t)$ for the interval $|t|\max\left\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\right\} \le \frac{1}{8}$, we have

$$|\delta(t)| \;\le\; |\varepsilon(t)| + |r(t)| \;\le\; 4^{s-2} L_s |t|^{3(s-2)} + e^{s-2} L_s \max\left\{|t|^s, |t|^{3(s-2)}\right\}.$$

Here $4^{-2} + e^{-2} < 0.2$, and we arrive at the first conclusion for $p = 0$.

In fact, one can a little sharpen the bound on $|r(z)|$, by noting that

$$\sum \prod_{l=3}^{m} \frac{1}{k_{l-2}!}\left(\frac{1}{l}\right)^{k_{l-2}} \;\le\; \exp\left\{\sum_{l=3}^{m} \frac{1}{l}\right\} - 1 \;\le\; e^{\log m - \log 2} - 1 \;=\; \frac{m-2}{2}.$$

Hence

$$|r(z)| \;\le\; \frac{s-2}{2} L_s \max\left\{|z|^s, |z|^{3(s-2)}\right\}.$$

This bound can be used for the remaining cases $1 \le p \le [s]$. One may apply the Cauchy contour integral formula to get that

$$|r^{(p)}(t)| \;\le\; \frac{p!}{\rho^p} \max_{|z-it|=\rho} |r(z)|.$$

Let us choose $\rho = \frac{1}{2}\,|t|$ and use the assumption $|t| \max\left\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\right\} \le \frac{1}{8}$. On the circle $|z - it| = \rho$ it is necessary that $|z| \le \frac{3}{2}\,|t|$ and thus $|z| \max\left\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\right\} \le \frac{1}{4}$. Hence, we may apply the previous step with bounding $r(z)$ which was made under the weaker assumption $L_s^{\frac{1}{s-2}}|z| \le 1$. This gives

$$\begin{aligned}|r^{(p)}(z)| &\le \frac{p!}{|0.5\,t|^p}\,\frac{s-2}{2}\,L_s \max\left\{|z|^s, |z|^{3(s-2)}\right\}\\ &\le \frac{p!}{|t|^p}\,2^s\,\frac{s-2}{2}\,L_s \max\left\{|1.5\,t|^s, |1.5\,t|^{3(s-2)}\right\}.\end{aligned}$$

This yields

$$|r^{(p)}(t)| \le 2p!\,6.75^{s-2}(s-2)\,L_s \max\left\{|t|^{s-p}, |t|^{3(s-2)-p}\right\}.$$

Again, by Lemma 11.2 with its function $\varepsilon(t)$,

$$\begin{aligned}|\delta(t)| &\le |\varepsilon(t)| + |r(t)|\\ &\le 4^{-2}\,p!\,4^s L_s |t|^{3(s-2)} + 2p!\,6.75^{s-2}\,(s-2)\,L_s \max\left\{|t|^{s-p}, |t|^{3(s-2)-p}\right\}.\end{aligned}$$

Here $2 \cdot 6.75^{s-2}(s-2) \le \frac{2}{e \log \frac{7}{6.75}}\, 7^{s-2} < 0.413 \cdot 7^s$, and then we arrive at the desired conclusion. $\square$

Corollary 12.2 *Let $L_s < \infty$ ($s \ge 3$). In the interval $|t| \max\left\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\right\} \le \frac{1}{8}$, we have $|P_m(it)| \le 0.1$. Moreover, for all $p = 1, \dots, [s]$,*

$$\frac{1}{p!}\left|\frac{d^p}{dt^p}\,P_m(it)\right| \le 1.4\,|t|^{-p}.$$

Proof First consider the case $p = 0$. By Lemma 11.1, $|Q_m(it)| \le 0.007$, which implies, using the second estimate of Lemma 11.1 and our assumption,

$$\left|e^{Q_m(it)} - 1\right| \le \frac{e^{0.007} - 1}{0.007}\,|Q_m(it)| \le 1.004 \cdot 0.42\, L_s^{\frac{1}{s-2}}\,|t|^3 \le 1.004 \cdot 0.42 \cdot \frac{1}{8^3} < 0.001.$$

In addition, by Proposition 12.1 (the obtained bound in case $p = 0$),

$$|\delta(t)| \le 0.2 \cdot 4^s L_s \max\left\{|t|^s, |t|^{3(s-2)}\right\}.$$

By the assumption, $L_s|t|^s \le \frac{1}{|8t|^{s-2}}\,|t|^s = \frac{1}{8^{s-2}}\,t^2$ and $L_s|t|^s \le \frac{1}{|8t|^{3(s-2)}}\,|t|^s = \frac{1}{8^{3(s-2)}}\,t^{6-2s}$. Both estimates yield $L_s|t|^s \le 8^{-s}$. Since also $L_s|t|^{3(s-2)} \le 8^{-s}$, we have

$$L_s \max\left\{|t|^s, |t|^{3(s-2)}\right\} \le 8^{-s},$$

so

$$|P_m(it)| \le \left|e^{Q_m(it)} - 1\right| + |\delta(t)| \le 0.001 + 0.2 \cdot \left(\frac{4}{8}\right)^s < 0.1,$$

which proves the corollary in this particular case.

Now, let $1 \le p \le [s]$. Combining Lemma 11.3 and Proposition 12.1, we have, using the previous step and the assumption $s \ge 3$:

$$\begin{aligned}\left|\frac{d^p}{dt^p}\,P_m(it)\right| &\le \left|\frac{d^p}{dt^p}\,e^{Q_m(it)}\right| + \left|\frac{d^p}{dt^p}\,\delta(t)\right| \\ &\le 1.01\,p!\,|t|^{-p} + 0.5\,p!\,7^s L_s \max\left\{|t|^{s-p}, |t|^{3(s-2)-p}\right\} \\ &\le p!\,|t|^{-p}\Big[\,1.01 + 0.5\left(\frac{7}{8}\right)^s\Big] \le p!\,|t|^{-p}\Big[\,1.01 + 0.5\left(\frac{7}{8}\right)^3\Big].\end{aligned}$$

□

13 Corrected Normal Approximation on Moderate Intervals

We are now prepared to prove several assertions about the corrected normal approximation for the characteristic function $f_n(t)$ of the sum $S_n = X_1 + \cdots + X_n$ of independent random variables X_k. As usual, we assume that $\mathbb{E}X_k = 0$, $\mathbb{E}X_k^2 = \sigma_k^2$ ($\sigma_k \ge 0$) with $\sum_{k=1}^n \sigma_k^2 = 1$. Recall that Lyapunov's coefficients are defined by

$$L_s = \sum_{k=1}^n \mathbb{E}\,|X_k|^s, \qquad s \ge 2.$$

As before, we write $s = m + \alpha$, where m is integer and $0 < \alpha \le 1$. The range $2 < s \le 3$ was considered in Propositions 9.1–9.2, so our main concern will be the case $s > 3$. As a preliminary step, let us prove the following statement, including the value $s = 3$ (a limit case).

Lemma 13.1 *Let $L_s < \infty$. In the interval $|t| \max\left\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\right\} \le \frac{1}{8}$, we have*

$$f_n(t) = e^{-t^2/2}\left(1 + P_m(it) + r(t)\right) \tag{13.1}$$

with

$$\left|\frac{d^p}{dt^p}\,r(t)\right| \le C_s L_s \max\left\{|t|^{s-p}, |t|^{3(s-2)-p}\right\}, \qquad p = 0, 1, \dots, [s], \tag{13.2}$$

where one may take $C_s = 0.4 \cdot 4^s$ in case $p = 0$ and $C_s = 1.8 \cdot 7^s$ for $1 \le p \le [s]$.

Proof Combining Proposition 12.1 and Corollary 12.2 with Proposition 8.2, we may write

$$f_n(t) \ = \ e^{-t^2/2}\, e^{Q_m(it)}\, (1+\varepsilon(t)) \ = \ e^{-t^2/2}\, (1+P_m(it)+\delta(t))\, (1+\varepsilon(t))$$

with

$$|\delta(t)| \le 0.2\cdot 4^s L_s \max\big\{|t|^s, |t|^{3(s-2)}\big\}, \qquad |\varepsilon(t)| \le 2^s L_s\, |t|^s, \qquad |P_m(it)| \le 0.1,$$

where the second inequality was derived under the assumption that $L_s^{\frac{1}{s}}\,|t| \le \frac{1}{8}$. It is fulfilled, since in general $L_s^{\frac{1}{s}} \le \max\big\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\big\}$. In particular, we get $L_s|t|^s \le 8^{-s}$, so $|\varepsilon(t)| \le 4^{-s}$.

Since

$$r(t) = (1+P_m(it))\varepsilon(t) + \delta(t)(1+\varepsilon(t)),$$

we obtain that

$$\begin{aligned} |r(t)| &\le 1.1\cdot 2^s L_s|t|^s + 0.2\cdot 4^s L_s \max\big\{|t|^s, |t|^{3(s-2)}\big\}\cdot(1+4^{-s}) \\ &\le 4^s L_s \max\big\{|t|^s, |t|^{3(s-2)}\big\}\Big[\,1.1\cdot\Big(\frac{2}{4}\Big)^s + 0.2 + 0.2\cdot 4^{-s}\Big]. \end{aligned}$$

The expression in square brackets does not exceed $1.1\cdot(\frac{2}{4})^3 + 0.2 + 0.2\cdot 4^{-3} < 0.4$, which proves the assertion in case $p = 0$.

Now, let us turn to the derivatives of order $p = 1, \dots, [s]$ and apply other bounds given in Proposition 12.1, Corollary 12.2, and Proposition 8.2,

$$\begin{aligned} \frac{1}{p!}\,|\delta^{(p)}(t)| &\le 0.5\cdot 7^s L_s \max\big\{|t|^{s-p}, |t|^{3(s-2)-p}\big\}, \\ \frac{1}{p!}\,|\varepsilon^{(p)}(t)| &\le 2\cdot\frac{4^s}{s}\, L_s\, |t|^{s-p}, \\ \frac{1}{p!}\,|P_m^{(p)}(it)| &\le 1.4\,|t|^{-p} \end{aligned}$$

(which remain to hold in case $p = 0$ as well). Differentiating the product $P_m(it)\varepsilon(t)$ according to the Newton binomial formula, let us write

$$\big((1+P_m(it))\cdot\varepsilon(t)\big)^{(p)} = \sum_{k=0}^{p} \frac{p!}{k!\,(p-k)!}\,\big(1+P_m(it)\big)^{(k)}\varepsilon(t)^{(p-k)}.$$

Applying the above estimates, we then get

$$\begin{aligned}\Big|\big((1+P_m(it))\cdot\varepsilon(t)\big)^{(p)}\Big| &\leq \sum_{k=0}^{p}\frac{p!}{k!\,(p-k)!}\,k!\;1.4\cdot|t|^{-k}\cdot(p-k)!\,\frac{2\cdot 4^s}{s}\,L_s|t|^{s-(p-k)}\\ &= 2.8\,\frac{(p+1)!}{s}\,4^s\,L_s|t|^{s-p}\\ &\leq 2.8\cdot p!\,4^s\,L_s|t|^{s-p}.\end{aligned}$$

To derive a similar bound for the product $\delta(t)\varepsilon(t)$, we use $L_s|t|^{3(s-2)} \leq 8^{-3(s-2)}$ together with $L_s|t|^s \leq 8^{-s}$. Then, the estimate on the p-th derivative of δ implies

$$|\delta^{(p)}(t)| \;\leq\; p!\,0.5\cdot 7^s\,8^{-s}\,|t|^{-p} \;\leq\; 0.4\,p!\,|t|^{-p}.$$

Hence, again according to the binomial formula,

$$\begin{aligned}\Big|\big(\delta(t)\varepsilon(t)\big)^{(p)}\Big| &\leq \sum_{k=0}^{p}\frac{p!}{k!\,(p-k)!}\,0.4\,k!\,|t|^{-k}\cdot(p-k)!\,4^sL_s\,|t|^{s-(p-k)}\\ &= 0.4\,(p+1)!\,4^s\,L_s|t|^{s-p}.\end{aligned}$$

Collecting these estimates, we obtain that

$$\begin{aligned}|r^{(p)}(t)| &\leq \Big|(P_m(it)\varepsilon(t))^{(p)}\Big| + \Big|(\delta(t)\varepsilon(t))^{(p)}\Big| + \Big|\varepsilon^{(p)}(t)\Big| + \Big|\delta^{(p)}(t)\Big|\\ &\leq p!\,\big(2.8\cdot 4^s + 0.4\,(p+1)\,4^s + 4^s + 0.5\cdot 7^s\big)\,L_s\max\big\{|t|^{s-p},\,|t|^{3(s-2)-p}\big\}.\end{aligned}$$

Here, since the function $t \to te^{-\beta t}$ is decreasing for $t > 1/\beta$ ($\beta > 0$), we have

$$(p+1)\,4^s \leq \frac{7}{4}\,(s+1)\left(\frac{4}{7}\right)^{s+1}7^s \leq 4\left(\frac{4}{7}\right)^3 7^s < 0.75\cdot 7^s.$$

In addition, $4^s = (\frac{4}{7})^s\,7^s < 0.2\cdot 7^s$. So the expression in the brackets in front of L_s is smaller than $(2.8\cdot 0.2 + 0.4\cdot 0.75 + 0.2 + 0.7)\,7^s < 1.8\cdot 7^s$. □

In the representation for $f_n(t)$ in (13.1), one can take the term $r(t)$ out of the brackets, and then we get a more convenient form (at the expense of a larger power of t). Thus, put

$$g_m(t) = e^{-t^2/2}\,(1 + P_m(it)),$$

which serves as the corrected normal "characteristic" function. For the first values of m, one may recall the formulas for P_m at the end of Section 10, which give $g_2(t) = e^{-t^2/2}$,

$$\begin{aligned} g_3(t) &= e^{-t^2/2}\Big(1+\gamma_3\frac{(it)^3}{3!}\Big),\\ g_4(t) &= e^{-t^2/2}\Big(1+\gamma_3\frac{(it)^3}{3!}+\gamma_4\frac{(it)^4}{4!}+\gamma_3^2\,\frac{(it)^6}{2!\,3!^2}\Big),\\ g_5(t) &= e^{-t^2/2}\Big(\gamma_3\frac{(it)^3}{3!}+\gamma_4\frac{(it)^4}{4!}+\gamma_5\frac{(it)^5}{5!}+\gamma_3^2\,\frac{(it)^6}{2!\,3!^2}+\gamma_3^3\,\frac{(it)^9}{3!\,3!^3}\Big). \end{aligned}$$

Proposition 13.2 *Let $L_s < \infty$ $(s \geq 3)$. In the interval $|t|\max\big\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\big\} \leq \frac{1}{8}$, for every $p = 0, 1, \dots, [s]$,*

$$\Big|\frac{d^p}{dt^p}\big(f_n(t) - g_m(t)\big)\Big| \leq C_s L_s \max\big\{|t|^{s-p}, |t|^{3(s-2)+p}\big\}\,e^{-t^2/2}, \tag{13.3}$$

where one may take $C_s = 0.5\cdot 4^s$ in case $p = 0$ and $C_s = 6\cdot 8^s$ for $1 \leq p \leq [s]$.

Proof Using the remainder term in (13.1), consider the function

$$R(t) \equiv f_n(t) - e^{-t^2/2}\,(1 + P_m(it)) = e^{-t^2/2}\,r(t).$$

In case $p = 0$, (13.2) gives the bound

$$|r(t)| \leq 0.5\cdot 4^s L_s \max\big\{|t|^s, |t|^{3(s-2)}\big\}.$$

Hence, the same uniform bound holds for $R(t)$ as well.

Turning to the derivatives, we use the bounds

$$\frac{1}{p!}\,|r^{(p)}(t)| \leq 1.8\cdot 7^s L_s \max\big\{|t|^{s-p}, |t|^{3(s-2)-p}\big\}$$

together with $|g^{(p)}(t)| \leq p!\,\max\{1, |t|^p\}\,g(t)$ for the Gaussian function $g(t) = e^{-t^2/2}$ (cf. (1.8)). Differentiating the product according to the binomial formula,

$$R^{(p)}(t) = \sum_{k=0}^{p} \frac{p!}{k!\,(p-k)!}\,g^{(p-k)}(t) r^{(k)}(t),$$

we therefore obtain that the absolute value of the above sum is bounded by

$$g(t)\sum_{k=0}^{p}\frac{p!}{k!\,(p-k)!}\,(p-k)!\,\max\big\{1, |t|^{p-k}\big\}\cdot k!\;1.8\cdot 7^s L_s \max\big\{|t|^{s-k}, |t|^{3(s-2)-k}\big\}$$

$$\leq 1.8\cdot 7^s p!\,L_s g(t)\sum_{k=0}^{p}\max\big\{1, |t|^{p-k}\big\}\max\big\{|t|^{s-k}, |t|^{3(s-2)-k}\big\}$$

$$\leq \; 1.8 \cdot 7^s p!\, L_s g(t)\, (p+1)\; \max\left\{|t|^{s-p}, |t|^{3(s-2)+p}\right\}.$$

Here

$$(p+1)\, 7^s \leq \frac{8}{7}\, (s+1) \left(\frac{7}{8}\right)^{s+1} 8^s \leq \frac{8}{7}\, \frac{1}{e \log \frac{8}{7}}\, 8^s \leq 3.15 \cdot 8^s,$$

while $3.15 \cdot 1.8 < 5.7$. □

Remarks In the literature one can find different variations of the inequality (13.3). For integer values $s = m + 1$ and for $p = 0$, it was proved by Statulevičius, cf. [St1, St2] (with a similar behavior of the constants). A somewhat more complicated formulation describing the multidimensional expansion was given by Bikjalis [Bi2] (in the same situation).

14 Signed Measures μ_m Associated with g_m

Once it is observed that the characteristic function $f_n(t)$ of S_n is close on a relatively long interval to the corrected normal "characteristic function" $g_m(t) = e^{-t^2/2}\,(1 + P_m(it))$, it is reasonable to believe that in some sense the distribution of S_n is close to the signed measure μ_m, whose Fourier-Stieltjes transform is exactly $g_m(t)$, that is, with

$$\int_{-\infty}^{\infty} e^{itx}\, d\mu_m(x) = g_m(t), \qquad t \in \mathbb{R}.$$

In order to describe μ_m, let us recall the Chebyshev-Hermite polynomials

$$H_k(x) = (-1)^k\, (e^{-x^2/2})^{(k)}\, e^{x^2/2}, \qquad k = 0, 1, 2, \dots \quad (x \in \mathbb{R}),$$

or equivalently, $\varphi^{(k)}(x) = (-1)^k H_k(x)\varphi(x)$ in terms of the normal density $\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$. Each H_k is a polynomial of degree k with leading coefficient 1. For example,

$$H_0(x) = 1, \qquad H_2(x) = x^2 - 1, \qquad H_4(x) = x^4 - 6x^2 + 3,$$

$$H_1(x) = x, \qquad H_3(x) = x^3 - 3x, \qquad H_5(x) = x^5 - 10x^3 + 15x,$$

$$H_6(x) = x^6 - 15x^4 + 45x^2 - 15,$$

and so on. These polynomials are orthogonal on the real line with weight $\varphi(x)$ and form a complete orthogonal system in the Hilbert space $L^2(\mathbb{R}, \varphi(x)\, dx)$. By the repeated integration by parts (with $t \neq 0$),

$$e^{-t^2/2} = \int_{-\infty}^{\infty} e^{itx}\varphi(x)\, dx = \frac{1}{-it} \int_{-\infty}^{\infty} e^{itx}\varphi'(x)\, dx = \frac{1}{(-it)^k} \int_{-\infty}^{\infty} e^{itx}\varphi^{(k)}(x)\, dx.$$

In other words, we have the identity $\int_{-\infty}^{\infty} e^{itx}\, H_k(x)\varphi(x)\, dx \ = \ (it)^k\, e^{-t^2/2}$. Equivalently, using the inverse Fourier transform, one may write

$$H_k(x)\,\varphi(x) \ = \ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}\, (it)^p\, e^{-t^2/2}\, dt,$$

which may be taken as another definition of H_k.

Returning to Definition 10.1, we therefore obtain:

Proposition 14.1 *Let $L_s < \infty$ for $s = m+\alpha$ with an integer $m \geq 2$ and $0 < \alpha \leq 1$. The measure μ_m with Fourier-Stieltjes transform $g_m(t) \ = \ e^{-t^2/2}\,(1 \ + \ P_m(it))$ has density*

$$\varphi_m(x) = \varphi(x) + \varphi(x) \sum \frac{1}{k_1!\dots k_{m-2}!} \Big(\frac{\gamma_3}{3!}\Big)^{k_1} \dots \Big(\frac{\gamma_m}{m!}\Big)^{k_{m-2}}\, H_k(x),$$

where $k = 3k_1 + \dots + mk_{m-2}$ and where the summation runs over all collections of non-negative integers $(k_1, \dots, k_{m-2})$ that are not all zero and such that $k_1 + 2k_2 + \dots + (m-2)k_{m-2} \leq m-2$.

Recall that the cumulants γ_p of S_n are well defined for $p \ = \ 1, \dots, m$ and also for $p \ = \ m+1$ when s is integer. However, in this case γ_{m+1} is not present in the construction of φ_m. By the definition, if $2 < s \leq 3$, the above sum is empty, that is, $\varphi_2 = \varphi$.

In a more compact form, one may write $\varphi_m(x) = \varphi(x)(1 + R_m(x))$, where R_m is a certain polynomial of degree at most $3(m-2)$, defined by

$$R_m(x) = \sum \frac{1}{k_1!\dots k_{m-2}!} \Big(\frac{\gamma_3}{3!}\Big)^{k_1} \dots \Big(\frac{\gamma_m}{m!}\Big)^{k_{m-2}}\, H_k(x),$$

where $k = 3k_1 + \dots + mk_{m-2}$ and the summation is as before. For $m = 3$, we have $R_3(x) \ = \ \frac{\gamma_3}{3!}\, H_3(x) \ = \ \frac{\gamma_3}{3!}\, (x^3 - 3x)$, while for $m = 4$,

$$\begin{aligned} R_4(x) \ &= \ \frac{\gamma_3}{3!}\, H_3(x) + \frac{\gamma_4}{4!}\, H_4(x) + \frac{\gamma_3^2}{2!\,3!^2}\, H_6(x) \\ &= \ \frac{\gamma_3}{3!}\, (x^3 - 3x) + \frac{\gamma_4}{4!}\, (x^4 - 6x^2 + 3) + \frac{\gamma_3^2}{2!\,3!^2}\, (x^6 - 15x^4 + 45x^2 - 15). \end{aligned}$$

Correspondingly, for $m = 5$,

$$R_5(x) \ = \ \frac{\gamma_3}{3!}\, H_3(x) + \frac{\gamma_4}{4!}\, H_4(x) + \frac{\gamma_5}{5!}\, H_5(x) + \frac{\gamma_3^2}{2!\,3!^2}\, H_6(x) + \frac{\gamma_3^3}{3!\,3!^3}\, H_9(x).$$

Let us briefly describe a few basic properties of the measures μ_m.

Proposition 14.2 *The moments of S_n and μ_m coincide up to order m, that is,*

$$f_n^{(p)}(0) = g_m^{(p)}(0), \qquad p = 0, 1, \dots, m.$$

In particular, $\mu_m(\mathbb{R}) = \int_{-\infty}^{\infty} \varphi_m(x)\, dx = 1$.

The latter immediately follows from the Fourier transform formula

$$\int_{-\infty}^{\infty} e^{itx} \varphi_m(x)\, dx = g_m(t) = e^{-t^2/2}\,(1 + P_m(it)),$$

applied at $t = 0$. The more general assertion immediately follows from Proposition 13.2, which gives $|f_n^{(p)}(t) - g_m^{(p)}(t)| = O(|t|^{s-p})$ as $t \to 0$.

Proposition 14.3 *If $L_s < \infty$ for $s = m + \alpha$ with $m \geq 2$ integer and $0 < \alpha \leq 1$, then the measure μ_m has a total variation norm satisfying*

$$\big|\|\mu_m\|_{\mathrm{TV}} - 1\big| \leq m\sqrt{(3(m-2))!}\, \max\{L_s^{\frac{1}{s-2}}, L_s^{\frac{m-2}{s-2}}\}. \tag{14.1}$$

In addition,

$$\int_{-\infty}^{\infty} |x|^s\, |\mu_m(dx)| \leq s^{2s} \max\{L_s, 1\}. \tag{14.2}$$

Proof In the definition of P_m, the tuples $(k_1, \dots, k_{m-2})$ participating in the sum satisfy $1 \leq d \leq m-2$, where $d = k_1 + 2k_2 + \cdots + (m-2)k_{m-2}$. Thus (cf. (10.1)),

$$\begin{aligned}
\Big|\Big(\frac{\gamma_3}{3!}\Big)^{k_1} \cdots \Big(\frac{\gamma_m}{m!}\Big)^{k_{m-2}}\Big| &\leq \frac{1}{3^{k_1} \dots m^{k_{m-2}}}\, L_s^{\frac{d}{s-2}} \\
&\leq \frac{1}{3^{k_1} \dots m^{k_{m-2}}}\, \max\{L_s^{\frac{1}{s-2}}, L_s^{\frac{m-2}{s-2}}\} \;\leq\; \frac{1}{3^{k_1} \dots m^{k_{m-2}}}\, \max\{L_s, 1\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\big|\|\mu_m\|_{\mathrm{TV}} - 1\big| &= \int_{-\infty}^{\infty} |R_m(x)|\, \varphi(x)\, dx \\
&\leq \max\{L_s^{\frac{1}{s-2}}, L_s^{\frac{m-2}{s-2}}\} \sum \frac{1}{k_1! \dots k_{m-2}!}\, \frac{1}{3^{k_1} \dots m^{k_{m-2}}} \int_{-\infty}^{\infty} |H_k(x)|\, \varphi(x)\, dx,
\end{aligned}$$

where $k = 3k_1 + \cdots + mk_{m-2}$ (which may vary from 3 to $3(m-2)$). Let Z be a random variable with the standard normal distribution. As is well known,

$$\int_{-\infty}^{\infty} H_k(x)^2\, \varphi(x)\, dx = \mathbb{E}\, H_k(Z)^2 = k!$$

Hence, by the Cauchy inequality,

$$\int_{-\infty}^{\infty} |H_k(x)|\, \varphi(x)\, dx = \mathbb{E}\, |H_k(Z)| \le \sqrt{k!} \le \sqrt{(3(m-2))!}$$

implying that

$$\big|\|\mu_m\|_{\rm TV} - 1\big| \le \sqrt{(3(m-2))!}\, \max\{L_s, L_s^{(m-2)/(s-2)}\} \sum \frac{1}{k_1! \dots k_{m-2}!}\, \frac{1}{3^{k_1} \dots m^{k_{m-2}}}.$$

The latter sum does not exceed $e^{1/3} \dots e^{1/m} < m$, cf. (10.2), and we obtain (14.1).

Let us now turn to the second assertion. If $m = 2$, then $\varphi_m = \varphi$ and μ_m is the standard Gaussian measure on the real line. In this case,

$$\int_{-\infty}^{\infty} |x|^s\, |\mu_m(dx)| = \mathbb{E}\, |Z|^s = \frac{2^{s/2}}{\sqrt{\pi}}\, \Gamma\Big(\frac{s+1}{2}\Big) \le \frac{2^{3/2}}{\sqrt{\pi}}\, \Gamma(2) < 1.6 < s^{2s}.$$

In case $m \ge 3$, again by the Cauchy inequality, for the same value of k as before, we have

$$\begin{aligned}\int_{-\infty}^{\infty} |x|^s\, |H_k(x)|\, \varphi(x)\, dx = \mathbb{E}\, |Z|^s\, |H_k(Z)| \ &\le \ \sqrt{\mathbb{E}\, |Z|^{2s}}\sqrt{k!} \\ &\le \ \sqrt{\mathbb{E}\, |Z|^{2s}}\sqrt{(3(m-2))!}\end{aligned}$$

Hence, applying once more the inequality (10.2) together with the last bound on the product of the cumulants, we obtain that

$$\begin{aligned}\int_{-\infty}^{\infty} |x|^s\, |\mu_m(dx)| &\le \int_{-\infty}^{\infty} |x|^s\, \varphi(x)\, dx + \int_{-\infty}^{\infty} |x|^s\, |R_m(x)|\, \varphi(x)\, dx \\ &\le \mathbb{E}\, |Z|^s + \sqrt{\mathbb{E}\, |Z|^{2s}}\, m \sqrt{(3(m-2))!}\, \max\{L_s, 1\} \\ &\le 2\sqrt{\mathbb{E}\, |Z|^{2s}}\, m \sqrt{(3(m-2))!}\, \max\{L_s, 1\}.\end{aligned}$$

To simplify the right-hand side, one may use

$$(3(m-2))! \le \Gamma(3s-5) = \frac{\Gamma(3s+1)}{3s\,(3s-1)(3s-2)(3s-3)(3s-4)(3s-5)}.$$

Since $3s - 1 \ge \frac{8}{3}\, s$, $3s - 2 \ge \frac{7}{3}\, s$, $3s - 3 \ge 6$, $3s - 4 \ge 5$, $3s - 5 \ge 4$, we have $\Gamma(3s-5) \le \frac{1}{280\, s^3}\, \Gamma(3s+1)$ and thus

$$\Big(\frac{1}{\max(L_s, 1)} \int_{-\infty}^{\infty} |x|^s\, |\mu(dx)|\Big)^2$$

$$\le 4\,\mathbb{E}\,|Z|^{2s}\,\frac{m^2}{280\,s^3}\,\Gamma(3s+1)\ \le\ \frac{1}{70\,s}\,\mathbb{E}\,|Z|^{2s}\,\Gamma(3s+1)$$
$$= \frac{1}{70\,s}\,\frac{2^s}{\sqrt{\pi}}\,\Gamma\Big(s+\frac{1}{2}\Big)\,\Gamma(3s+1)\ <\ \frac{1}{70\,s}\,\frac{2^s}{\sqrt{\pi}}\,\Gamma(s+1)\,\Gamma(3s+1).$$

By Stirling's formula, $\Gamma(x+1) \le 2\,(\frac{x}{e})^x\sqrt{2\pi x}$ $(x \ge 3)$, which allows us to bound the above right-hand side by

$$\frac{1}{70\,s}\,\frac{2^s}{\sqrt{\pi}}\cdot 2\Big(\frac{s}{e}\Big)^s\sqrt{2\pi s}\cdot 2\Big(\frac{3s}{e}\Big)^{3s}\sqrt{6\pi s}\ =\ \frac{2\sqrt{12\,\pi}}{35}\,\Big(\frac{54}{e^4}\Big)^s s^{4s}\ <\ s^{4s}.$$

$\square$

As a consequence, one can complement Proposition 14.3 with the following statement which is of a special interest when L_s is large (since in that case the interval of approximation in this proposition is getting small).

Corollary 14.4 *Let $L_s < \infty$ for $s = m + \alpha$ with $m \ge 2$ integer and $0 < \alpha \le 1$. Then, for all $t \in \mathbb{R}$ and $p = 0, 1, \dots, [s]$,*

$$\Big|\frac{d^p}{dt^p}\,\big(f_n(t) - g_m(t)\big)\Big|\ \le\ 4s^{2s}\,\max\{L_s, 1\}\frac{|t|^{s-p}}{([s]-p)!}.$$

Proof Let P_n denote the distribution of S_n. By Proposition 5.2 applied to the signed measure $\mu = P_n - \mu_m$, we have

$$\Big|\frac{d^p}{dt^p}\,\big(f_n(t) - g_m(t)\big)\Big|\ \le\ 2C_s\,\frac{|t|^{s-p}}{(m-p)!},\quad \text{where}\ \ C_s = \int_{-\infty}^{\infty}|x|^s\,|P_n(dx) - \mu_m(dx)|.$$

Here

$$C_s \le \int_{-\infty}^{\infty}|x|^s\,P_n(dx) + \int_{-\infty}^{\infty}|x|^s|\mu_m(dx)| = \mathbb{E}\,|S_n|^s + \int_{-\infty}^{\infty}|x|^s|\mu_m(dx)|.$$

The last integral may be estimated with the help of the bound (14.2), while the s-th absolute moment of S_n is estimated with the help of Rosenthal's inequality $\mathbb{E}\,|S_n|^s \le (2s)^s\,\max\{L_s, 1\}$, cf. (7.3). Since $(2s)^s \le s^{2s}$, we get $C_s \le 2s^{2s}$. $\square$

PART IV. Corrected normal approximation on long intervals

15 Upper Bounds for Characteristic Functions f_n

Let $X_1, \dots, X_n$ be independent random variables with $\mathbb{E}X_k = 0$, $\mathbb{E}X_k^2 = \sigma_k^2$ $(\sigma_k \geq 0)$, assuming that $\sum_{k=1}^n \sigma_k^2 = 1$. Recall that

$$L_3 = \sum_{k=1}^n \mathbb{E}\,|X_k|^3.$$

On long intervals of the t-axis, we are aimed to derive upper bounds on the absolute value of the characteristic function $f_n(t) = \mathbb{E}\,e^{itS_n}$ of the sum $S_n = X_1 + \dots + X_n$. Assume that X_k have finite 3-rd moments, and put $\beta_{3,k} = \mathbb{E}|X_k|^3$. We will need:

Lemma 15.1 *Let X be a random variable with characteristic function $v(t)$. If $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2$, $\mathbb{E}\,|X|^3 = \beta_3 < \infty$, then for all $t \in \mathbb{R}$,*

$$|v(t)| \leq e^{-\frac{1}{2}\sigma^2 t^2 + \frac{1}{3}\beta_3 |t|^3}.$$

In addition, if $\beta_s = \mathbb{E}\,|X|^s$ is finite for $s \geq 3$, then for all $p = 1, \dots, [s]$,

$$|v^{(p)}(t)| \ \leq \ e^{1/6}\,\beta_{p^*} \max\{1, |t|\}\, e^{-\frac{1}{2}\sigma^2 t^2 + \frac{1}{3}\beta_3 |t|^3}, \qquad p^* = \max\{p, 2\}.$$

Proof Let X' be an independent copy of X. Since X has mean zero, $\mathbb{E}\,|X - X'|^3 \leq 4\beta_3$, cf. [B-RR], Lemma 8.8. Hence, by Taylor's expansion, for any t real,

$$|v(t)|^2 = \mathbb{E}\,e^{it(X-X')} = 1 - \sigma^2 t^2 + \frac{4\theta}{3!}\,\beta_3|t|^3 \leq \exp\Big\{ - \sigma^2 t^2 + \frac{4\theta}{3!}\,\beta_3|t|^3\Big\}$$

with some $\theta = \theta(t)$ such that $|\theta| \leq 1$. The first inequality now easily follows.

Since $|v''(t)| \leq \sigma^2$ and $v'(0) = 0$, we also have $|v'(t)| \leq \sigma^2|t|$. On the other hand, putting $x = \sigma|t|$ and using $\beta \geq \sigma^3$, we have

$$-\frac{1}{2}\,\sigma^2 t^2 + \frac{1}{3}\,\beta_3|t|^3 \geq -\frac{1}{2}\,x^2 + \frac{1}{3}\,x^3 \geq -\frac{1}{6} \qquad (x \geq 0).$$

This proves the second inequality of the lemma in case $p = 1$. If $p \geq 2$, then we only need to apply $|v^{(p)}(t)| \leq \beta_p$. □

Denoting by v_k the characteristic function of X_k, by the first inequality of Lemma 15.1, $|v_k(t)| \leq \exp\{-\frac{1}{2}\,\sigma_k^2 t^2 + \frac{1}{3}\,\beta_{3,k}|t|^3\}$. Multiplying these inequalities, we get

$$|f_n(t)| \leq \exp\Big\{ - \frac{1}{2}\,t^2 + \frac{1}{3}\,L_3|t|^3\Big\}.$$

If $|t| \leq \frac{1}{L_3}$, then $L_3|t|^3 \leq t^2$ for $|t| \leq \frac{1}{L_3}$. Hence, the above bound yields:

Proposition 15.2 *We have* $|f_n(t)| \le e^{-t^2/6}$ *whenever* $|t| \le \frac{1}{L_3}$.

One can sharpen the statement of Proposition 15.2 by developing Taylor's expansion for $v_k(t)$, rather than for $|v_k(t)|^2$. By Taylor's integral formula,

$$v_k(t) = 1 - \frac{\sigma_k^2 t^2}{2} + \frac{1}{2} \int_0^t v_k'''(\tau)(t-\tau)^2 \, d\tau,$$

so $\left| v_k(t) - (1 - \frac{\sigma_k^2 t^2}{2}) \right| \le \frac{\beta_{3,k}}{6} |t|^3$. Here the left-hand side dominates $|v_k(t)| - (1 - \frac{\sigma_k^2 t^2}{2})$ in case $\sigma_k |t| \le \sqrt{2}$, and then we obtain that

$$|v_k(t)| \le 1 - \frac{\sigma_k^2 t^2}{2} + \frac{\beta_{3,k} |t|^3}{6} \le \exp\Big\{ - \frac{\sigma_k^2 t^2}{2} + \frac{\beta_{3,k} |t|^3}{6} \Big\}.$$

Multiplying these inequalities, we get:

Proposition 15.3 *If* $\max_k \sigma_k |t| \le \sqrt{2}$, *we have*

$$|f_n(t)| \le \exp\Big\{ - \frac{t^2}{2} + \frac{L_3 |t|^3}{6} \Big\}.$$

Hence, if additionally $|t| \le \frac{1}{L_3}$, *then* $|f_n(t)| \le e^{-t^2/3}$.

This statement has an advantage over Proposition 15.2 in case of i.i.d. summands.

Now let us consider the case of the finite L_s with $2 < s \le 3$ and define $\beta_{s,k} = \mathbb{E}\, |X_k|^s$. Here is an adaptation of Lemma 15.1.

Lemma 15.4 *Let* X *be a random variable with characteristic function* $v(t)$. *If* $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2$, $\mathbb{E}\,|X|^s = \beta_s < \infty$ *for* $2 < s \le 3$, *then, for all* $t \in \mathbb{R}$,

$$|v(t)| \le e^{-\frac{1}{2}\sigma^2 t^2 + 2\beta_s |t|^s}.$$

In addition,

$$|v'(t)| \le e^{1/24}\, \sigma^2 |t|\, e^{-\frac{1}{2}\sigma^2 t^2 + 2\beta_s |t|^s}, \qquad |v''(t)| \le e^{1/24}\, \sigma^2 e^{-\frac{1}{2}\sigma^2 t^2 + 2\beta_s |t|^s}.$$

Proof Let X' be an independent copy of X. Then $\mathrm{Var}(X - X') = 2\sigma^2$. Write

$$\begin{aligned} |X - X'|^s &= (X - X')^2\, |X - X'|^{s-2} \\ &\le (X - X')^2 \left(|X|^{s-2} + |X'|^{s-2} \right) = (X^2 - 2XX' + X') \left(|X|^{s-2} + |X'|^{s-2} \right), \end{aligned}$$

implying that

$$\begin{aligned} \mathbb{E}\,|X - X'|^s &\le \mathbb{E}\,|X|^s + \mathbb{E}\,|X'|^s + \mathbb{E}X^2\, \mathbb{E}\,|X'|^{s-2} + \mathbb{E}X'^2\, \mathbb{E}\,|X|^{s-2} \\ &= 2\, \mathbb{E}\,|X|^s + 2\, \mathbb{E}X^2\, \mathbb{E}\,|X|^{s-2}. \end{aligned}$$

Here $\mathbb{E}X^2 \le \beta_s^{2/s}$ and $\mathbb{E}\,|X|^{s-2} \le \beta_s^{(s-2)/s}$, so that we obtain $\mathbb{E}\,|X - X'|^s \le 4\beta_s$.

Now, by Proposition 5.1 with $p = 0$, $m = 2$, applied to $X - X'$,

$$|v(t)|^2 = \mathbb{E}\,e^{it(X-X')} = 1 - \sigma^2 t^2 + \delta(t), \qquad |\delta(t)| \le 4\beta_s |t|^s.$$

Hence, for any t real,

$$|v(t)|^2 \le 1 - \sigma^2 t^2 + 4\beta_s |t|^s \le \exp\big\{-\sigma^2 t^2 + 4\beta_s |t|^s\big\},$$

proving the first inequality. Since $|v''(t)| \le \sigma^2$ and $v'(0) = 0$, we also have $|v'(t)| \le \sigma^2 |t|$, $|v''(t)| \le \sigma^2$. On the other hand, putting $x = \sigma|t|$ and using $\beta_s \ge \sigma^s$, we have

$$-\frac{1}{2}\,\sigma^2 t^2 + 2\beta_s |t|^s \ge -\frac{1}{2}\,x^2 + 2x^s = \psi(x).$$

On the positive half-axis the function ψ attains minimum at the point $x_s = (2s)^{-\frac{1}{s-2}}$, at which

$$\psi(x_s) \;=\; -\frac{1}{2}\,(2s)^{-\frac{2}{s-2}} + 2\,(2s)^{-\frac{s}{s-2}} \;=\; -\frac{s-2}{2s}\left(\frac{1}{2s}\right)^{\frac{2}{s-2}} \;\ge\; -\frac{1}{24}.$$

□

Now, returning to the random variables X_k, by the first inequality of this lemma, we have $|v_k(t)| \le \exp\big\{-\frac{1}{2}\,\sigma_k^2 t^2 + 2\beta_{s,k}|t|^s\big\}$. Multiplying them, we get $|f_n(t)| \le \exp\{-\frac{t^2}{2}\,(1 - 4L_s |t|^{s-2})\}$, which yields:

Proposition 15.4 *If* $2 < s \le 3$, *then* $|f_n(t)| \le e^{-t^2/6}$ *in the interval* $|t| \le (6L_s)^{-\frac{1}{s-2}}$.

Remarks The first inequality in Lemma 15.1 first appeared apparently in the work by Zolotarev [Z1]. Later in [Z2] he sharpened this bound to

$$\log|v(t)| \le -\frac{1}{2}\,\sigma^2 t^2 + 2\kappa_3\,\beta_3 |t|^3, \qquad \kappa_3 = \sup_{x>0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^3} = 0.099\ldots$$

Further refinements are due to Prawitz [Pr1, Pr2]. Sharper forms of Lemma 15.4, including s-dependent constants in front of $|t|^s$ for the values $2 < s \le 3$, were studied by Ushakov and Shevtsova, cf. [U], [Sh2].

16 Bounds on the Derivatives of Characteristic Functions

Keeping notations of the previous section together with basic assumptions on the random variables X_k's, here we extend upper bounds on the characteristic function $f_n(t) = \mathbb{E}\,e^{itS_n}$ to its derivatives up to order $[s]$. Put $p^* = \max(p, 2)$.

Proposition 16.1 *Let $L_s < \infty$, for some $s \geq 3$. Then, for all $p = 0, \dots, [s]$,*

$$\left| \frac{d^p}{dt^p} f_n(t) \right| \leq 2.03^p \, p! \, \max\{L_{p^*}, 1\} \, \max\{1, |t|^p\} \, e^{-t^2/6}, \qquad \text{if } \; |t| \leq \frac{1}{L_3}. \tag{16.1}$$

Proof The case $p = 0$ follows from Proposition 15.2. For $p \geq 1$, denote by $v_k(t)$ the characteristic functions of X_k. We use the polynomial formula

$$f_n^{(p)}(t) = \sum \binom{p}{q_1 \; \dots \; q_n} v_1^{(q_1)}(t) \dots v_n^{(q_n)}(t)$$

with summation running over all integers $q_k \geq 0$ such that $q_1 + \dots + q_n = p$. By Lemma 15.1,

$$|v_k(t)| \leq e^{-\frac{1}{2}\sigma_k^2 t^2 + \frac{1}{3}\beta_{3,k}|t|^3},$$

$$|v_k^{(q_k)}(t)| \leq e^{1/6} \, \beta_{q_k^*,k} \, \max\{1, |t|\} \, e^{-\frac{1}{2}\sigma_k^2 t^2 + \frac{1}{3}\beta_{3,k}|t|^3}, \qquad q_k \geq 1,$$

where $\beta_{q,k} = \mathbb{E}\,|X_k|^q$ and $q_k^* = \max\{q_k, 2\}$. Applying these inequalities and noting that the number

$$l = \text{card}\{k \leq n : q_k \geq 1\}$$

is smaller than or equal to p, we get

$$\prod_{k=0}^{n} |v_k^{(q_k)}(t)| \leq e^{p/6} \max\{1, |t|^p\} \, e^{-\frac{1}{2}t^2 + \frac{1}{3}L_3|t|^3} \, \beta_{q_1^*,1} \dots \beta_{q_n^*,n}.$$

Write $(q_1, \dots, q_n) = (0, \dots, q_{k_1}, \dots, q_{k_l}, \dots, 0)$, specifying all indexes k for which $q_k \geq 1$. Put $p_1 = q_{k_1}, \dots, p_l = q_{k_l}$. Thus, $p_j \geq 1$, $p_1 + \dots + p_l = p$, so $1 \leq l \leq p$, and the above bound takes the form

$$\prod_{k=0}^{n} |v_k^{(q_k)}(t)| \leq e^{p/6} \max\{1, |t|^p\} \, e^{-\frac{1}{2}t^2 + \frac{1}{3}L_3|t|^3} \, \beta_{p_1^*,k_1} \dots \beta_{p_l^*,k_l}.$$

Using it in the polynomial formula and performing summation over all k_j's, we arrive at

$$|f_n^{(p)}(t)| \leq e^{p/6} \max\{1, |t|^p\} \, e^{-\frac{1}{2}t^2 + \frac{1}{3}L_3|t|^3} \, \widetilde{L}_p$$

with

$$\widetilde{L}_p = \sum \binom{p}{p_1 \; \dots \; p_l} L_{p_1^*} \dots L_{p_l^*},$$

where the sum runs over all integers $l = 1, \dots, p$ and $p_1, \dots, p_l \geq 1$ such that $p_1 + \dots + p_l = p$.

Clearly, $\widetilde{L}_1 = 1$ and $\widetilde{L}_2 = 2$. If $p \geq 3$, using the property that the function $q \to L_q^{1/(q-2)}$ is not decreasing in $q > 2$ (Proposition 7.1), we get

$$L_{p_1^*} \dots L_{p_l^*} = \prod_{j:\, p_j \geq 2} L_{p_j} \leq \prod_{j:\, p_j \geq 2} L_p^{\frac{p_j - 2}{p-2}} = L_p^\nu.$$

Here

$$(p-2)\nu = \sum_{j=1}^{l} (p_j - 2)\, 1_{\{p_j \geq 2\}} = p - 2l + \sum_{j:\, p_j = 1} 1 \leq p - 2$$

with the last inequality holding for $l \geq 2$. Also, when $l = 1$, necessarily $\sum_{j:p_j=1} 1 = 0$, so $\nu \leq 1$ in all cases. But then $L_p^\nu \leq \max\{L_p, 1\}$, which implies

$$\begin{aligned} \widetilde{L}_p &\leq \max\{L_p, 1\} \sum_{l=1}^{p} \sum_{p_1 + \dots + p_l = p} \binom{p}{p_1 \ \dots \ p_l} \\ &\leq \max\{L_p, 1\}\, p! \prod_{l=1}^{p} \sum_{p_l=1}^{\infty} \frac{1}{p_l!} \leq \max\{L_p, 1\}\, (e-1)^p\, p! \end{aligned}$$

This inequality remains to hold for $p = 1$ and $p = 2$. Thus, for all $p \geq 1$,

$$|f_n^{(p)}(t)| \leq \big((e-1)\, e^{1/6}\big)^p p! \max\{L_{p^*}, 1\} \max\{1, |t|^p\}\, e^{-\frac{1}{2} t^2 + \frac{1}{3} L_3 |t|^3}.$$

Here $(e-1)e^{1/6} < 2.03$. Also, if $|t| \leq \frac{1}{L_3}$, then $L_3 |t|^3 \leq t^2$. □

Let us now turn to the case $2 < s < 3$ with finite Lyapunov coefficient L_s rather than L_3. In terms of the characteristic functions $v_k(t)$, the first derivative of $f_n(t)$ is just the sum

$$f_n'(t) = \sum_{k=1}^{n} v_1(t) \dots v_{k-1}(t)\, v_k'(t)\, v_{k+1}(t) \dots v_n(t).$$

Here, by Lemma 15.4, the k-th term is dominated by $e^{1/24}\, \sigma_k^2 |t|\, e^{-\frac{1}{2} t^2 + 2L_s |t|^s}$. Performing summation over all $k \leq n$, we then arrive at

$$|f_n'(t)| \leq e^{1/24}\, |t|\, e^{-\frac{1}{2} t^2 + 2L_s |t|^s}.$$

Now, let us turn to the second derivative. Assuming that $n \geq 2$, first write

$$f_n''(t) = \sum_{k=1}^{n} v_1(t) \dots v_{k-1}(t)\, v_k''(t)\, v_{k+1}(t) \dots v_n(t)$$
$$+ 2 \sum_{1 \leq k < l \leq n} v_1(t) \dots v_{k-1}(t)\, v_k'(t)\, v_{k+1}(t) \dots v_{l-1}(t)\, v_l'(t)\, v_{l+1}(t) \dots v_n(t).$$

Again by Lemma 15.4, we get

$$|v_1(t) \dots v_{k-1}(t)\, v_k''(t)\, v_{k+1}(t) \dots v_n(t)| \leq e^{1/24}\, \sigma_k^2\, e^{-\frac{1}{2}t^2 + 2L_s|t|^s}$$

and

$$|v_1(t) \dots v_{k-1}(t)\, v_k'(t)\, v_{k+1}(t) \dots v_{l-1}(t)\, v_l'(t)\, v_{l+1}(t) \dots v_n(t)|$$
$$\leq\ e^{1/12}\, \sigma_k^2 \sigma_l^2\, t^2\, e^{-\frac{1}{2}t^2 + 2L_s|t|^s}.$$

Performing summation in the representation for $f_n''(t)$ we arrive at

$$|f_n''(t)|\ \leq\ \big(e^{1/24} + e^{1/12}\, t^2\big)\, e^{-\frac{1}{2}t^2 + 2L_s|t|^s}.$$

If $n = 1$, the estimate is simplified to $|f_1''(t)| \leq e^{1/24}\, e^{-\frac{1}{2}t^2 + 2L_s|t|^s}$. One can summarize.

Proposition 16.2 *If* $2 < s < 3$, *then in the interval* $|t| \leq (6L_s)^{-\frac{1}{s-2}}$,

$$|f_n(t)| \leq e^{-t^2/6}, \qquad |f_n'(t)| \leq e^{1/24}\, |t|\, e^{-t^2/6}, \qquad |f_n''(t)| \leq e^{1/12}(1 + t^2)\, e^{-t^2/6}.$$

17 Upper Bounds for Approximating Functions $g_m(t)$

Our next step is to get bounds, similar to the ones in Sections 15–16, for the corrected normal "characteristic function"

$$g_m(t) = e^{-t^2/2}\, (1 + P_m(it))$$

with large values of $|t|$, more precisely – outside the interval of Proposition 13.2.

Proposition 17.1 *Let* $s \geq 3$. *In the region* $|t|\ \max\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\} \geq \frac{1}{8}$, *we have*

$$|g_m(t)|\ \leq\ (142\, s)^{3s/2}\, L_s\, e^{-t^2/8}. \tag{17.1}$$

Moreover, for every $p = 1, 2, \dots, [s]$,

$$|g_m^{(p)}(t)| \;\leq\; (573\, s)^{2s}\, L_s\, e^{-t^2/8}. \tag{17.2}$$

Recall that, for real values $s = m + \alpha$, where $m \geq 2$ is integer and $0 < \alpha \leq 1$,

$$P_m(it) = \sum \frac{1}{k_1! \dots k_{m-2}!} \left(\frac{\gamma_3}{3!}\right)^{k_1} \dots \left(\frac{\gamma_m}{m!}\right)^{k_{m-2}} (it)^k,$$

where the summation runs over all collections of non-negative integers $(k_1, \dots, k_{m-2})$ that are not all zero and such that

$$k \equiv 3k_1 + \dots + mk_{m-2}, \qquad d \equiv k_1 + 2k_2 + \dots + (m-2)k_{m-2} \leq m - 2.$$

Note that all tuples that are involved satisfy $1 \leq d \leq s-2$ and $1 \leq k \leq 3d \leq 3(s-2)$.

Proof of Proposition 17.1 We use the bound (10.1), implying that, for all complex t,

$$|P_m(it)| \;\leq\; \sum \frac{1}{k_1! \dots k_{m-2}!}\, \frac{1}{3^{k_1} \dots m^{k_{m-2}}}\, L_s^{\frac{d}{s-2}}\, |t|^k.$$

If $L_s \geq 1$, then $L_s^{\frac{d}{s-2}} \leq L_s$. In this case, using a simple inequality

$$x^\beta e^{-x} \leq (\beta e^{-1})^\beta \qquad (x, \beta \geq 0) \tag{17.3}$$

together with the property $k \leq 3(s-2)$, we have

$$|t|^k\, e^{-3t^2/8} \leq \left(\frac{8k}{3e}\right)^{k/2} \leq \left(\frac{8s}{e}\right)^{\frac{3}{2}\,(s-2)} < (3s)^{\frac{3}{2}\,(s-2)}.$$

Hence $L_s^{\frac{d}{s-2}}\, |t|^k\, e^{-t^2/2} \;\leq\; (3s)^{\frac{3}{2}\,(s-2)}\, L_s\, e^{-t^2/8}$. Using the inequality (10.2), we then get

$$|g_m(t)| \leq (1 + |P_m(it)|)\, e^{-t^2/2} \;\leq\; m(3s)^{\frac{3(s-2)}{2}}\, L_s\, e^{-t^2/8} \;\leq\; (3s)^{3s/2}\, L_s\, e^{-t^2/8},$$

which provides the desired estimate (17.1).

In the (main) case $L_s \leq 1$, it will be sufficient to bound the products $L_s^{\frac{d}{s-2}-1}\, |t|^k\, e^{-3t^2/8}$ by the s-dependent constants uniformly for all admissible tuples. Put $x = L_s^{-\frac{1}{s-2}}$. Using the hypothesis $|t| \geq \frac{1}{8}\, x^{1/3}$, let us rewrite every such product and then estimate it as follows:

$$L_s^{\frac{d}{s-2}-1}\, |t|^k\, e^{-3t^2/8} = x^{(s-2)-d}\, e^{-t^2/4} \cdot |t|^k\, e^{-t^2/8}$$

$$\leq x^{(s-2)-d}\, e^{-\frac{1}{256}x^{2/3}} \cdot |t|^k\, e^{-t^2/8}$$
$$= (256\, y)^{\frac{3}{2}((s-2)-d)}\, e^{-y} \cdot (8u)^{k/2}\, e^{-u},$$

where we changed the variables $x = (256\, y)^{3/2}$, $t = (8u)^{1/2}$. Next, again we apply inequality (17.3), which allows us to bound the last expression by

$$\Big(256 \cdot \frac{3}{2e}\,(s-2-d)\Big)^{\frac{3}{2}(s-2-d)} \cdot \Big(\frac{8k}{2e}\Big)^{k/2} \leq \Big(\frac{384\, s}{e}\Big)^{\frac{3}{2}(s-2-d)} \cdot \Big(\frac{12\, s}{e}\Big)^{k/2}$$
$$\leq \Big(\frac{384\, s}{e}\Big)^{\frac{1}{2}(3(s-2-d)+k)}.$$

Here $3(s-2-d)+k = 3(s-2)-(3d-k) \leq 3(s-2)$. Hence, the last quantity may further be estimated by $\big(\frac{384\, s}{e}\big)^{\frac{3(s-2)}{2}} < (142\, s)^{\frac{3(s-2)}{2}}$, so

$$L_s^{\frac{d}{s-2}}\, |t|^k\, e^{-t^2/2} \leq (142\, s)^{\frac{3(s-2)}{2}}\, L_s\, e^{-t^2/8}.$$

This inequality remains to hold, when all $k_j = 0$ as well. Thus, similarly to the previous case,

$$(1 + |P_m(it)|)\, e^{-t^2/2} \;\leq\; m(142\, s)^{\frac{3(s-2)}{2}}\, L_s\, e^{-t^2/8} \;\leq\; (142\, s)^{3s/2}\, L_s\, e^{-t^2/8},$$

proving the first part of the proposition, i.e. for $p = 0$.

To treat the case of derivatives of an arbitrary order $p \geq 1$, one may use the property that g_m is an entire function and apply Cauchy's contour integral formula. This would reduce our task to bounding $|g_m|$ in a strip of the complex plane. Indeed, first consider the functions of the complex variable

$$R_k(z) = z^k\, e^{-z^2/2}, \qquad z = t + u, \;\; (t \neq 0 \text{ real}), \;\; |u| \leq \frac{|t|}{4} \;(u \text{ complex}).$$

We have $|z| \leq \frac{5}{4}\,|t|$ and $\mathrm{Re}(z^2) \geq t^2 - 2\,|t|\,|u| - |u|^2 \geq \frac{7}{16}\,t^2$, implying that

$$|R_k(z)| = |z|^k\, e^{-\mathrm{Re}(z^2)/2} \leq \Big(\frac{5}{4}\,|t|\Big)^k\, e^{-7t^2/32}.$$

For any $\rho > 0$, by Cauchy's integral formula, $|R_k^{(p)}(t)| \leq p!\, \rho^{-p}\, \max_{|z-t|=\rho} |R_k(z)|$. Choosing $\rho = \frac{|t|}{4}$ and applying the constraints $p \leq s+1$, $k \leq 3(s-2)$, we get

$$|R_k^{(p)}(t)| \;\leq\; p!\, \Big(\frac{4}{|t|}\Big)^p \cdot \Big(\frac{5}{4}\,|t|\Big)^k\, e^{-7t^2/32} \;\leq\; p!\, 4^{s+1} \Big(\frac{5}{4}\Big)^{3(s-2)} \cdot |t|^{k-p}\, e^{-7t^2/32}. \tag{17.4}$$

Case 1. First assume that $k \geq p$.

If $L_s \leq 1$, putting $x = L_s^{-\frac{1}{s-2}}$ as before and using the hypothesis $|t| \geq \frac{1}{8}\,x^{1/3}$, we have:

$$\begin{aligned}
p!\, L_s^{\frac{d}{s-2}-1}\, |t|^{k-p}\, e^{-3t^2/32} &= p!\, x^{(s-2)-d}\, e^{-t^2/16} \cdot |t|^{k-p}\, e^{-t^2/32} \\
&\leq p!\, x^{(s-2)-d}\, e^{-\frac{1}{8^2\cdot 16}\, x^{2/3}} \cdot |t|^{k-p}\, e^{-t^2/32} \\
&= p!\, (8^2 \cdot 16\, y)^{\frac{3}{2}\,((s-2)-d)}\, e^{-y} \cdot (32\, u)^{\frac{1}{2}\,(k-p)}\, e^{-u}.
\end{aligned}$$

Again using the general inequality (17.3), one can bound the last expression by

$$\begin{aligned}
&p!\, \Big(8^2 \cdot 16 \cdot \frac{3}{2e}\,(s-2-d)\Big)^{\frac{3}{2}\,(s-2-d)} \cdot \Big(32\, \frac{k-p}{2e}\Big)^{\frac{1}{2}\,(k-p)} \\
&\leq p!\, (566\, s)^{\frac{3}{2}\,((s-2)-d)} \cdot \Big(\frac{48s}{e}\Big)^{\frac{1}{2}\,(k-p)} \\
&\leq s^p\, (566\, s)^{\frac{1}{2}\,(3(s-2-d)+(k-p))},
\end{aligned}$$

where we applied elementary relations $p! \leq m^p \leq s^p$ for the values $p \leq m+1$ on the last step. Also note that

$$3(s-2-d) + (k-p) = 3(s-2) - (3d-k) - p \leq 3(s-2) - p.$$

Hence,

$$\begin{aligned}
s^p\, (566\, s)^{\frac{1}{2}\,(3(s-2-d)+(k-p))} &\leq 566^{\frac{3}{2}\,(s-2)}\, s^{\frac{1}{2}\,(3(s-2)+p)} \\
&\leq 116^{2(s-2)}\, s^{2s-2} \ < \ (116\, s)^{2s-2},
\end{aligned}$$

and thus

$$p!\, L_s^{\frac{d}{s-2}-1}\, |t|^{k-p}\, e^{-3t^2/32} \leq (116\, s)^{2s-2}. \tag{17.5}$$

If $L_s \geq 1$, the argument is similar and leads to a better constant. Since now $|t| \geq \frac{1}{8}\,x$,

$$\begin{aligned}
p!\, L_s^{\frac{d}{s-2}-1}\, |t|^{k-p}\, e^{-3t^2/32} &= p!\, x^{(s-2)-d}\, e^{-t^2/16} \cdot |t|^{k-p}\, e^{-t^2/32} \\
&\leq p!\, x^{(s-2)-d}\, e^{-\frac{1}{8^2\cdot 16}\, x^{2}} \cdot |t|^{k-p}\, e^{-t^2/32} \\
&= p!\, (8^2 \cdot 16\, y)^{\frac{1}{2}\,((s-2)-d)}\, e^{-y} \cdot (32\, u)^{\frac{1}{2}\,(k-p)}\, e^{-u}.
\end{aligned}$$

The last expression is bounded by

$$
\begin{aligned}
p! \left(8^2 \cdot 16 \cdot \frac{1}{2e}\,(s-2-d)\right)^{\frac{1}{2}\,(s-2-d)} &\cdot \left(32\,\frac{k-p}{2e}\right)^{\frac{1}{2}\,(k-p)} \\
&\leq p!\,(189\,s)^{\frac{1}{2}\,((s-2)-d)} \cdot \left(\frac{48s}{e}\right)^{\frac{1}{2}\,(k-p)} \\
&\leq s^p\,(189\,s)^{\frac{1}{2}\,((s-2-d)+(k-p))}.
\end{aligned}
$$

Replacing here $s-2-d$ with the larger value $3(s-2-d)$, we return to the previous step with constant 189 in place of 566. So, the bound (17.5) remains to hold in this case as well.

Case 2. Assume that $k < p$ and $L_s \leq 1$. In this case, the function $|t|^{k-p}\,e^{-3t^2/32}$ is decreasing in $|t|$. Using again $|t| \geq \frac{1}{8}\,x^{1/3}$ with $x = L_s^{-\frac{1}{s-2}}$, we have, by (17.3), for any $\beta \geq 0$,

$$
\begin{aligned}
p!\,L_s^{\frac{d}{s-2}-1}\,|t|^{k-p}\,e^{-3t^2/32} &\leq p!\,x^{(s-2)-d} \left(\frac{1}{8}\,x^{1/3}\right)^{k-p} e^{-\frac{3}{8^2\cdot 32}\,x^{2/3}} \\
&\leq p!\,8^s\,x^{(s-2)-d+\frac{1}{3}\,(k-p)}\,e^{-\frac{3}{8^2\cdot 32}\,x^{2/3}} \\
&= p!\,8^s \left(\frac{8^2\cdot 32\,\beta}{3e}\right)^{\beta} x^{(s-2)-d+\frac{1}{3}\,(k-p)-\frac{2}{3}\,\beta}.
\end{aligned}
$$

Here we choose β such that the power of x would be zero, that is, $\beta = \frac{3}{2}\,(s-2-d) + \frac{1}{2}\,(k-p)$. Let us verify that this number is indeed non-negative, that is, $(3d-k)+p \leq 3(s-2)$. This is obvious, when all $k_j = 0$. From the definition, it also follows that, when at least one $k_j > 0$,

$$
3d-k = 2\sum_{j=1}^{m-2}(j-1)k_j = 2d - 2\sum_{j=1}^{m-2}k_j \leq 2(m-2)-2.
$$

If $p \leq m$, we conclude that $(3d-k)+p \leq 2(m-2)-2+m = 3(m-2) < 3(s-2)$, which was required. If $s = m+1$ is integer, and $p = m+1$, we also have

$$
(3d-k)+p \leq 2(m-2)-2+(m+1) = 3m-5 < 3(s-2).
$$

Thus, one may use the chosen value of β. Since $\beta = \frac{1}{2}\,(3(s-2)-(3d-k)-p) \leq \frac{3(s-2)-p}{2}$, we then get that

$$
\begin{aligned}
p!\,L_s^{\frac{d}{s-2}-1}\,|t|^{k-p}\,e^{-3t^2/32} &\leq p!\,8^s \left(\frac{8^2\cdot 32\,\beta}{3e}\right)^{\beta} \\
&\leq s^p\,8^s \left(\frac{8^2\cdot 32\,s}{2e}\right)^{\frac{3(s-2)-p}{2}}
\end{aligned}
$$

$$\leq 8^s \left(\frac{8^2 \cdot 32}{2e}\right)^{\frac{3(s-2)}{2}} \cdot s^{\frac{3(s-2)+p}{2}} < (242\, s)^{2s-2}.$$

Case 3. Assume that $k < p$ and $L_s \geq 1$, $|t| \geq 1$. In this case one may just write

$$p!\, L_s^{\frac{d}{s-2}-1}\, |t|^{k-p}\, e^{-3t^2/32} \leq p! \leq s^p \leq s^{2s-2}.$$

Thus, in all these three cases,

$$p!\, L_s^{\frac{d}{s-2}-1}\, |t|^{k-p}\, e^{-3t^2/32} \leq (242\, s)^{2s-2},$$

and therefore, according to (17.4),

$$\begin{aligned} |R_k^{(p)}(t)| &\leq 4^{s+1} \left(\frac{5}{4}\right)^{3(s-2)} e^{-t^2/8} \cdot p!\, |t|^{k-p}\, e^{-3t^2/32} \\ &\leq 4^{s+1} \left(\frac{5}{4}\right)^{3(s-2)} (242\, s)^{2s-2}\, L_s^{1-\frac{d}{s-2}}\, e^{-t^2/8} \\ &< 573^{2s}\, s^{2s-2}\, L_s^{1-\frac{d}{s-2}}\, e^{-t^2/8}. \end{aligned} \tag{17.6}$$

Case 4. Assume that $k < p$, $L_s \geq 1$ and $|t| \leq 1$.

Returning to the Cauchy integral formula, let us now choose $\rho = 1$. For $z = t+u$, $|u| \leq 1$ (u complex), we have $|z| \leq 2$ and $\mathrm{Re}(z^2) \geq t^2 - 2\,|t|\,|u| - |u|^2 \geq \frac{1}{2}\,t^2 - 3$. Hence

$$|R_k(z)| = |z|^k\, e^{-\mathrm{Re}(z^2)/2} \leq e^{-3}\, 2^k\, e^{-t^2/2} \leq e^{-3}\, 2^{s+1}\, e^{-t^2/2}$$

and

$$|R_k^{(p)}(t)| \leq p! \max_{|z-t|=1} |R_k(z)| \leq p!\, e^{-3}\, 2^{s+1}\, e^{-t^2/2} \leq e^{-3}\, (2s)^{2s-2}\, e^{-t^2/2}.$$

This is better than the bound (17.6) obtained for the previous cases (note that the above right-hand side may be multiplied by the factor $L_s^{1-\frac{d}{s-2}}$ which is larger than 1).

As result, in all cases,

$$|R_k^{(p)}(t)| \leq 573^{2s}\, s^{2s-2}\, L_s^{1-\frac{d}{s-2}}\, e^{-t^2/8},$$

so

$$|g_m^{(p)}(t)| \leq \sum \frac{1}{k_1! \dots k_{m-2}!} \left| \left(\frac{\gamma_3}{3!}\right)^{k_1} \dots \left(\frac{\gamma_m}{m!}\right)^{k_{m-2}} \right| |R_k^{(p)}(t)|$$

$$\leq \sum \frac{1}{k_1! \dots k_{m-2}!} \frac{1}{3^{k_1} \dots m^{k_{m-2}}} L_s^{\frac{d}{s-2}} \cdot 573^{2s} s^{2s-2} L_s^{1-\frac{d}{s-2}} e^{-t^2/8}$$
$$\leq m \cdot 573^{2s} s^{2s-2} L_s e^{-t^2/8}.$$

□

18 Approximation of f_n and Its Derivatives on Long Intervals

Again, let $X_1, \dots, X_n$ be independent random variables with $\mathbb{E}X_k = 0$, $\sigma_k^2 = \mathbb{E}X_n^2$ ($\sigma_k \geq 0$) such that $\sum_{k=1}^n \sigma_k^2 = 1$, and finite Lyapunov coefficient L_s. On a relatively long (moderate) interval I_s, Proposition 13.2 (for $s \geq 3$) and Propositions 9.1–9.2 (for $2 < s \leq 3$) provide an approximation for the characteristic function $f_n(t)$ of the sum $S_n = X_1 + \dots + X_n$ by the corrected normal "characteristic function"

$$g_m(t) = e^{-t^2/2}\,(1 + P_m(it)).$$

This approximation also includes closeness of the derivatives of f_n and g_m up to order $[s]$. On the other hand, according to Propositions 15.2 and 16.1–16.2, $f_n(t)$ and their derivatives are very small in absolute value outside the interval I_s, although still inside $|t| \leq \frac{1}{L_3}$ when $s \geq 3$. Since $g_m(t)$ is also small (section 17), one can enlarge the interval I_s and thus simplify these approximations at the expense of a constant in the exponent appearing in the bounds.

As before, let $s = m + \alpha$, where $m \geq 2$ is integer and $0 < \alpha \leq 1$.

Theorem 18.1 *Let $L_s < \infty$ for $s \geq 3$. In the interval $|t| \leq \frac{1}{L_3}$,*

$$\left|f_n(t) - g_m(t)\right| \;\leq\; (Cs)^{3s/2} L_s \min\left\{1, |t|^s\right\} e^{-t^2/8}. \tag{18.1}$$

Moreover, for all $p = 0, 1, \dots, [s]$,

$$\left|\frac{d^p}{dt^p}\,(f_n(t) - g_m(t))\right| \;\leq\; (Cs)^{3s} L_s \min\left\{1, |t|^{s-p}\right\} e^{-t^2/8}, \tag{18.2}$$

where C is an absolute constant. One may take $C = 990$ in (18.1) *and $C = 70$ in* (18.2).

Proof We distinguish between several cases.

Case 1a. Moderate interval $I_s : |t| \max\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\} \leq \frac{1}{8}$. By Proposition 13.2, in this interval

$$|f_n(t) - g_m(t)| \;\leq\; 4^s L_s \max\left\{|t|^s, |t|^{3(s-2)}\right\} e^{-t^2/2}.$$

If $|t| \leq 1$, the above maximum is equal to $|t|^s$, and we are done with $C = 4$. If $|t| \geq 1$, the above maximum is equal to $|t|^{3(s-2)}$, and then one may use a general inequality $x^\beta e^{-x} \leq (\frac{\beta}{e})^\beta$ $(x, \beta > 0)$. For $x = 3t^2/8$ it gives

$$|t|^{3(s-2)}\, e^{-3t^2/8} = \Big(\frac{8x}{3}\Big)^{\frac{3(s-2)}{2}}\, e^{-x} \leq \Big(\frac{4\,(s-2)}{e}\Big)^{\frac{3(s-2)}{2}} < (1.48\, s)^{3s/2},$$

so

$$|t|^{3(s-2)}\, e^{-t^2/8} = |t|^{3(s-2)}\, e^{-3t^2/8}\, e^{-t^2/8} < (1.48\, s)^{3s/2}\, e^{-t^2/8}.$$

Since also $4^s \leq 2.52^{3s/2}$ and $2.52 \cdot 1.48 < 4$, we conclude that

$$|f_n(t) - g_m(t)| \;\leq\; (4s)^{3s/2}\, L_s \min\{1, |t|^s\}\, e^{-t^2/8}, \qquad t \in I_s,$$

which is the required inequality (18.1) with $C = 4$.

This bound may serve as a simplified version of Proposition 13.2 in the case $p = 0$. This is achieved at the expense of a worse constant in the exponent, although it contains a much larger s-dependent factor in front of L_s.

Case 2a. Large region $I_s' : |t| \max\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\} \geq \frac{1}{8}$ with $1 \leq |t| \leq \frac{1}{L_3}$. In this case, we bound both $f_n(t)$ and $g_m(t)$ in absolute value by appropriate quantities.

First, we involve the bound of Proposition 15.2, $|f_n(t)| \leq e^{-t^2/6}$, which is valid for $|t| \leq \frac{1}{L_3}$, and derive an estimate of the form

$$e^{-t^2/24} \leq C_s L_s.$$

If $L_s \geq 8^{-3(s-2)}$, it holds with $C_s = 8^{3(s-2)}$. If $L_s \leq 8^{-3(s-2)}$, then necessarily $|t| \geq \frac{1}{8}\, L_s^{-\frac{1}{3(s-2)}}$, and therefore one may take

$$C_s = \frac{1}{L_s}\, \exp\Big\{-\frac{1}{24 \cdot 8^2}\, L_s^{-\frac{2}{3(s-2)}}\Big\}.$$

Putting $L_s^{-\frac{2}{3(s-2)}} = 1536\, x$, the right-hand side equals and may be bounded with the help of (17.3) by

$$(1536\, x)^{\frac{3(s-2)}{2}}\, e^{-x} \leq \Big(\frac{1536 \cdot 3(s-2)}{2e}\Big)^{\frac{3(s-2)}{2}} < (848\, s)^{3s/2}.$$

As a result, we arrive at the upper bound

$$|f_n(t)| \;\leq\; e^{-t^2/24} \cdot e^{-t^2/8} \;\leq\; (848\, s)^{3s/2}\, L_s \min\{1, |t|^s\}\, e^{-t^2/8}.$$

A similar bound also holds for the approximating function $g_m(it) = e^{-t^2/2} + P_m(it)e^{-t^2/2}$. Recall that, by Proposition 17.1, whenever $|t| \geq 1$,

$$|g_m(t)| \leq (142\, s)^{3s/2}\, L_s\, e^{-t^2/8} \leq (142\, s)^{3s/2}\, L_s\, \min\{1, |t|^s\}\, e^{-t^2/8},$$

implying

$$|f_n(t) - g_m(t)| \leq \big((848\, s)^{3s/2} + (142\, s)^{3s/2}\big)\, L_s \min\{1, |t|^s\}\, e^{-t^2/8}.$$

Since $s > 3$, the constant in front of L_s is smaller than $990^{3s/2}$.

Case 3a. Consider the region $I_s' \,:\, |t| \max\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\} \geq \frac{1}{8}$ with $|t| \leq \min\{1, \frac{1}{L_3}\}$. Necessarily $L_s \geq 8^{-3(s-2)}$, so $\max\{L_s, 1\} \leq 8^{3(s-2)} L_s$. Hence, by Corollary 14.4 with $p = 0$,

$$\begin{aligned}|f_n(t) - g_m(t)| &\leq 4s^{2s}\, 8^{3(s-2)} L_s \frac{|t|^s}{m!} \\ &\leq \frac{4}{8^6} \frac{s^{2s}}{(m/e)^m}\, 8^{3s}\, L_s |t|^s \leq \frac{4}{8^6} \frac{s^{2s}}{(s/2)^{s/2}}\, e^s\, 8^{3s}\, L_s |t|^s < \frac{1}{2}\, (158\, s)^{3s/2}\, L_s |t|^s.\end{aligned}$$

This implies (18.1), since $e^{-t^2/8} \geq e^{-1/8}$.

The first assertion (18.1) is thus proved, and we now extend this inequality to the case of derivatives, although with a different dependence of the constants in s indicated in (18.2). We distinguish between several cases in analogy with the previous steps.

Case 1b. By Proposition 13.2, in the interval I_s,

$$|f_n^{(p)}(t) - g_m^{(p)}(t)| \leq 6 \cdot 8^s L_s \max\left\{|t|^{s-p}, |t|^{3(s-2)+p}\right\} e^{-t^2/2}.$$

If $|t| \leq 1$, the above maximum is equal to $|t|^{s-p}$, and we are done.
If $|t| \geq 1$, the above maximum is equal to $|t|^{3(s-2)+p}$. Using once more (17.3) with $x = 3t^2/8$, we have

$$|t|^{3(s-2)+p}\, e^{-3t^2/8} = \Big(\frac{8x}{3}\Big)^{\frac{3(s-2)+p}{2}} e^{-x} \leq \Big(\frac{4\,(3(s-2)+p)}{3e}\Big)^{\frac{3(s-2)+p}{2}} < (2s)^{2s},$$

so

$$|t|^{3(s-2)}\, e^{-t^2/8} = |t|^{3(s-2)}\, e^{-3t^2/8}\, e^{-t^2/8} < (2s)^{2s}\, e^{-t^2/8}.$$

Since also $6 \cdot 8^s \leq 4^{2s}$, we conclude that

$$|f_n^{(p)}(t) - g_m^{(p)}(t)| \leq (8s)^{2s} L_s \min\{1, |t|^{s-p}\}\, e^{-t^2/8}, \qquad t \in I_s,$$

which implies the required inequality (18.2) with $C = 8$.

Case 2b. Large region I_s' with $1 \leq |t| \leq \frac{1}{L_3}$. Let us involve Proposition 16.1. Using $p! \leq s^s$ and $\max\{L_{p^*}, 1\} \leq \max\{L_s, 1\}$, the bound (16.1) of this proposition readily implies

$$|f_n^{(p)}(t)| \leq (2.03\, s)^s \max\{L_s, 1\}\, |t|^s\, e^{-t^2/6}, \qquad 1 \leq |t| \leq \frac{1}{L_3}.$$

Thus, we need to derive an estimate of the form

$$(2.03\, s)^s \max\{L_s, 1\}\, |t|^s\, e^{-t^2/24} \leq C_s L_s.$$

If $L_s \geq 8^{-3(s-2)}$, the latter inequality holds with

$$C_s = (2.03\, s)^s\, 8^{3(s-2)} \max_t |t|^s\, e^{-t^2/24} = (2.03\, s)^s\, 8^{3(s-2)} \Big(\frac{12s}{e}\Big)^{s/2} < (13s)^{3s}.$$

If $L_s \leq 8^{-3(s-2)}$, then necessarily $|t| \geq \frac{1}{8}\, L_s^{-\frac{1}{3(s-2)}}$, i.e., $\frac{1}{L_s} \leq (8t)^{3(s-2)}$. Hence,

$$\begin{aligned}
\frac{1}{L_s}\, (2.03\, s)^s\, |t|^s\, e^{-t^2/24} &\leq (8t)^{3(s-2)}\, (2.03\, s)^s\, |t|^s\, e^{-t^2/24} \\
&= 8^{-6}\, (8^3 \cdot 2.03\, s)^s\, (24y)^{2s}\, e^{-y} \\
&\leq 8^{-6}\, (8^3 \cdot 2.03\, s)^s \Big(\frac{48\, s}{e}\Big)^{2s} < 8^{-6}\, 69^{3s}.
\end{aligned}$$

As a result, we arrive at the upper bound

$$|f_n^{(p)}(t)| \leq (69\, s)^{3s}\, L_s \min\{1, |t|^s\}\, e^{-t^2/8}.$$

As we know, a better bound holds for the function $g_m(it) = e^{-t^2/2} + P_m(it)e^{-t^2/2}$. By Proposition 17.1, whenever $|t| \geq 1$,

$$|g_m^{(p)}(t)| \leq (573\, s)^{2s}\, L_s\, e^{-t^2/8} \leq (69\, s)^{3s}\, L_s \min\{1, |t|^s\}\, e^{-t^2/8},$$

implying

$$|f_n^{(p)}(t) - g_m^{(p)}(t)| \leq (1 + 8^{-6})\, (69\, s)^{3s}\, L_s \min\{1, |t|^s\}\, e^{-t^2/8}.$$

Since $s > 3$, the constant in front of L_s is smaller than 70^{3s}.

Case 3b. The region I_s' with $|t| \leq \min\{1, \frac{1}{L_3}\}$. Necessarily $L_s \geq 8^{-3(s-2)}$, so $\max\{L_s, 1\} \leq 8^{3(s-2)} L_s$. Hence, by Corollary 14.4, for all $p \leq [s]$,

$$|f_n^{(p)}(t) - g_m^{(p)}(t)| \leq 4s^{2s}\, 8^{3(s-2)} L_s\, |t|^{s-p}$$

$$\leq \frac{4}{8^6} s^{2s} 8^{3s} L_s |t|^{s-p} \leq (8s)^{3s} L_s |t|^{s-p}.$$

Clearly, this bound is better than what was obtained on the previous step. □

Finally, let us include an analog of Theorem 18.1 for the case $2 < s < 3$. The following statement can be proved with similar arguments on the basis of Propositions 9.1 and 16.2.

Theorem 18.2 *Let $L_s < \infty$ for $2 < s < 3$. In the interval $|t| \leq (6L_s)^{-\frac{1}{s-2}}$, we have*

$$\left| \frac{d^p}{dt^p} \left(f_n(t) - e^{-t^2/2} \right) \right| \leq C L_s \min\left\{1, |t|^{s-p}\right\} e^{-t^2/8}, \qquad p = 0, 1, 2,$$

where C is an absolute constant.

Remarks In the literature, inequalities similar to (18.1)–(18.2) can be found for integer values $s = m + 1 \geq 3$, often for identically distributed summands $X_k = \xi_k / \sqrt{n}$, only, when $L_s = \beta_s n^{-(n-2)/2}$, $\beta_s = \mathbb{E}\,|\xi_1|^s$. In the book by Petrov [P2], (18.2) is proved without the derivative of the maximal order $p = m + 1$ and with an indefinite constant C_s (cf. Lemma 4, p. 140, which is attributed to Osipov [O]). Bikjalis derived a more precise statement (cf. [Bi3]). In case $p = 0$, he proved that, in the interval $|t| \leq \frac{1}{10} \beta_s^{-\frac{1}{s-2}} \sqrt{n}$,

$$\left| f_n(t) - g_m(t) \right| \leq \frac{2^{s-1}}{0.99^s} \beta_s n^{-\frac{s-2}{2}} |t|^s e^{-t^2/4}, \tag{18.3}$$

while for $p = 1, \dots, s$, $|t| \leq \frac{1}{16e} \beta_s^{-\frac{1}{s-2}} \sqrt{n}$, we have

$$\left| \frac{d^p}{dt^p} \left(f_n(t) - g_m(t) \right) \right| \leq \frac{p!^2\, 64^{s+p-2}}{s-2} \beta_s n^{-\frac{s-2}{2}} |t|^{s-p} e^{-t^2/6}. \tag{18.4}$$

It is interesting that the right-hand side in (18.3) provides a sharper growth of the constant in s in comparison with (18.1). Similarly, for the critical value $p = s$, the right-hand side in (18.4) may be replaced with $(Cs)^{2s} L_s \min\{1, |t|^s\} e^{-t^2/8}$ which also gives some improvement over (18.2). On the other hand, inequalities (18.1)–(18.2) are applicable in the non-i.i.d. situation and for real values of s.

In the general non-i.i.d. case, some similar versions of (18.1) were studied in [Bi1, Bi2]. A variant of (18.2) can be found in the book by Bhattacharya and Ranga Rao [B-RR], who considered multidimensional summands. Their Theorem 9.9 covers the interval $|t| \leq c L_s^{-\frac{1}{s-2}}$, although it does not specify constants as functions of s. Note that the interval $|t| \leq 1/L_3$ as in Theorem 18.1 is longest possible (up to a universal factor), but we leave open the question on the worst growth rates of the s-dependent constants in such inequalities.

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