Remarks on the Growth of L^p-norms of Polynomials

S.G. Bobkov

Department of Mathematics, Syktyvkar University, 167001 Syktyvkar, Russia

Abstract. We study the behaviour of constants in Khinchine-Kahane-type inequalities for polynomials in random vectors which have logarithmically concave distributions.

It is well known that polynomials on $\mathbf{R}^n$ of bounded degree satisfy dimension free Khinchine-Kahane-type inequalities with respect to the uniform distribution on convex sets. The best known result, which is due to J. Bourgain [Bou] and which gave an affirmative answer to a conjecture of V. D. Milman, indicates the following: there exist universal constants $t_0 > 0$ and $c \in (0,1)$ such that, for every convex set $K \subset \mathbf{R}^n$ of volume one, every polynomial $f = f(x_1, \dots, x_n)$ of degree $d \geq 1$, we have a distribution inequality

$$\mu_K\{|f| \geq ||f||_1 t\} \leq \exp\{-t^{c/d}\}, \quad t \geq t_0, \tag{1}$$

where μ_K is the Lebesgue measure on K, and $||f||_1$ is L^1-norm of f with respect to μ_K. As usual, for $p > 0$, one denotes $||f||_p = (\int |f|^p \, d\mu)^{1/p}$ which also refers to some probability measure μ on a space where the function f is defined. The inequality (1) may also be written in terms of a suitable Orlicz norm. For $\alpha \geq 1$, set $\psi_\alpha(t) = \exp\{t^\alpha\} - 1$, $t \geq 0$, and introduce the associated norm

$$||f||_{\psi_\alpha} = \inf\left\{\lambda > 0 : \int \psi_\alpha(|f|/\lambda) \, d\mu \leq 1\right\}.$$

Then, (1) is equivalent to the inequality

$$||f||_{\psi_{c/d}} \leq C^d \, ||f||_1, \tag{2}$$

with some universal $c \in (0,1)$ and $C > 0$. In particular, this yields the equivalence between L^p and L^1-norms in the form of the Khinchine-Kahane-type inequality

$$||f||_p \leq C(d,p) \, ||f||_1, \tag{3}$$

where $C(d,p)$ depends on d and p, only. The case $d = 1$, with the best constant $c = 1$ in (2), was settled before by M. Gromov and V. D. Milman [G-M1] (cf. also [M-S]). It is therefore interesting to know whether or not the inequality

The work was partially supported by a grant of the Russian Foundation for Fundamental Research.

(2) with $c = 1$ holds true in the general case $d \geq 1$. The latter is equivalent to the statement that the inequality (3) holds in the range $p \geq 1$ with constants $C(d,p) = (Cdp)^d$, for some universal C. In this note, we would like to show, following R. Kannan, L. Lovász and M. Simonovits [K-L-S], a short proof and a refinement of such a statement involving more general classes of probability measures on $\mathbf{R}^n$.

Theorem 1. *With respect to an arbitrary log-concave probability measure μ on $\mathbf{R}^n$, for every polynomial f on $\mathbf{R}^n$ of degree $d \geq 1$, we have, for some universal C,*

$$\|f\|_{\psi_{1/d}} \leq C^d \, \|f\|_0. \tag{4}$$

Here, $\|f\|_0 = \lim_{p \to 0} \|f\|_p = \exp\left\{\int \log |f| \, d\mu\right\}$. The log-concavity of μ means (cf. [Bor]) that μ is supported by some affine subspace E of $\mathbf{R}^n$ where it has a logarithmically concave density with respect to Lebesgue measure on E, i.e., a density $u : E \to [0, +\infty)$ such that, for all $x, y \in E$, and all $t, s \geq 0$ with $t + s = 1$,

$$u(tx + sy) \geq u(x)^t u(y)^s.$$

As a main statement (Theorem 1.3 in [Bou]), J. Bourgain formulated the inequality (2) with, however, a constant C instead of C^d. As we believe, the power d was lost when deriving (2) from (1). That the power of a constant cannot be omitted can be seen on the example of the set $K = [0,1]$ with the function $f = 1$ and with growing d. On the other hand, the power can be hidden, if one wishes to rewrite for example the inequality (4) in the equivalent form as

$$\left\| \, |f|^{1/d} \, \right\|_{\psi_1} \leq C \left\| \, |f|^{1/d} \, \right\|_0$$

(with the same constant). This inequality is very much similar to what is known for norms on $\mathbf{R}^n$ in the place of $|f|^{1/d}$ (cf. [L]).

To prove (4), we use Theorem 2.7 from [K-L-S] which is based on the localization lemma of L. Lovász and M. Simonovits [L-S]. It reduces multidimensional Khinchine-Kahane-type inequalities with respect to log-concave measures to dimension one that was illustrated in [K-L-S] on the example of inequalities of the form (3) in the linear case $d = 1$. It was also mentioned there that the method applies as well to the general case $d \geq 1$ and allows one to recover Bourgain's theorem. This is completely true up to a remark that the one-dimensional case in (3) still requires and deserves a special careful consideration, since it determines behaviour of constants as a function of d and p. The localization lemma of [L-S] has also applications to other types of inequalities. In connection with isoperimetric inequalities on the sphere, a kind of localization technique was developed by M. Gromov and V. D. Milman in [G-M2], cf. also [A].

For the Gaussian measure on the real line $\mathbf{R}$, using expansions over Hermit polynomials, the inequality (3) with $p = 2$ was studied by Yu. V. Prokhorov

[P1]; he obtained as well similar inequalities when μ belongs to the family of $\Gamma(\alpha)$-distributions with the parameter α growing with the degree d, cf. [P2]. As shown in these works, in both cases, the constants $C(d,2)$ grow exponentially, and this fact cannot be recovered by an inequality such as (2) or (4). Combining Prokhorov's approach with the localization method, one can prove:

Theorem 2. *With respect to an arbitrary log-concave probability measure μ on $\mathbf{R}^n$, for every polynomial f on $\mathbf{R}^n$ of degree $d \geq 1$, the Khinchine-Kahane-type inequality* (3) *holds with*

$$C(d,p) = p^{Cd}, \quad p \geq 2,$$

where C is universal.

Consider the example of the exponential measure $\mu = \nu$ on $\mathbf{R}$ with density $\frac{d\nu(x)}{dx} = e^{-x}$, $x > 0$, and take $f(x) = x^d$. Then, $\|f\|_p = \Gamma(dp+1)^{1/p}$ and, by Stirling's formula,

$$c_1 \frac{p^d}{d^{(p-1)/(2p)}} \|f\|_1 \leq \|f\|_p \leq c_2 \frac{p^d}{d^{(p-1)/(2p)}} \|f\|_1,$$

for some $c_1, c_2 > 0$. Therefore, when p is fixed and d grows, the constant $C(d,p) = p^{Cd}$ gives a correct exponential rate of increase.

To compare with Theorem 1, note that, up to a universal constant, the inequality (4) is equivalent to

$$\|f\|_p \leq (Cdp)^d \|f\|_0, \quad p \geq 1/d. \tag{5}$$

Here, since we have replaced $\|f\|_1$ with a smaller quantity $\|f\|_0$, the constants have a different order. Again, for $f(x) = x^d$, we have $\|f\|_0 = \|x\|_0^d$ with respect to ν, so

$$(c_1 dp)^d \|f\|_0 \leq \|f\|_p \leq (c_2 dp)^d \|f\|_0,$$

for some $c_1, c_2 > 0$. One could also test sharpness of (5) on the example of the uniform distribution on the ℓ^1-ball in $\mathbf{R}^n$ for $f(x) = x_1^d$ and with growing dimension n.

Proof of Theorem 1. First let us comment on the one-dimensional case in (5). Motivated by the results of [P1][P2], inequalities of the form (5), with respect to an arbitrary probability measure μ on $\mathbf{R}$, were studied in [B-G]. As was observed there, given $p > 0$ and $d \geq 1$, the optimal constant $C = C(d,p;\mu)$ in

$$\|f\|_p \leq C \|f\|_0, \tag{6}$$

where f is an arbitrary polynomial of degree d on $\mathbf{R}$ with complex coefficients, is given by

$$C^{1/d} = \sup_{z \in \mathbf{C}} \frac{\|x - z\|_{dp}}{\|x - z\|_0}. \tag{7}$$

Since the argument is straightforward, let us recall it. A remarkable feature of the functional $||f||_0$ is its multiplicativity property: $||f_1 \dots f_d||_0 = ||f_1||_0 \dots ||f_d||_0$. Therefore, writing $f(x) = A(x - z_1) \dots (x - z_d)$ and applying Hölder's inequality, we get

$$||f||_p \leq A \prod_{k=1}^{d} ||x - z_k||_{pd} \leq CA \prod_{k=1}^{d} ||x - z_k||_0 = C||f||_0,$$

where C is defined according to (7). This proves (6) with this constant which cannot be improved as the example of the polynomials $f(x) = x - z$ shows.

It might be helpful to note that the sup in (7) can be restricted to the real line $\mathbf{R}$. Indeed, write $z = a + bi$ so that $|x - z|^2 = \xi + t$ with $\xi = (x - a)^2 \geq 0$ and $t = b^2 \geq 0$. As can easily be verified by differentiation, for any $p > q$, the function of the form $g(t) = ||\xi + t||_p / ||\xi + t||_q$ is non-increasing in $t \geq 0$, hence it is maximized at $t = 0$. Therefore, as a function of b, the value of

$$\frac{||x - z||_p}{||x - z||_q} = \frac{||\xi + t||_{p/2}^{1/2}}{||\xi + t||_{q/2}^{1/2}}$$

is maximized at $b = 0$. So, we may apply this observation with $q = 0$ to (7).

In the particular case, where μ is log-concave, the right hand side of (7) can be bounded by a quantity which is independent of μ and grows like Cdp. Indeed, after shifting, one needs to estimate an optimal constant C in

$$|| \, |x| \, ||_{dp} \leq C^{1/d} \, || \, |x| \, ||_0, \quad p \geq 1/d, \tag{8}$$

with respect to a log-concave measure on $\mathbf{R}$. The fact that such an inequality holds for an arbitrary log-concave measure μ on $\mathbf{R}^n$, and for an arbitrary norm $||x||$ instead of $|x|$ was established by R. Latala [L] (cf. also [B] and [Gu] for different proofs). More precisely, he showed that, for some universal C_1, we always have

$$|| \, ||x|| \, ||_1 \leq C_1 \, || \, ||x|| \, ||_0.$$

On the other hand, it had been known, as an application of Borell's lemma [Bor], that, for $p \geq 1$,

$$|| \, ||x|| \, ||_p \leq C_2 p \, || \, ||x|| \, ||_1.$$

These two inequalities give $|| \, ||x|| \, ||_p \leq C_0 p \, || \, ||x|| \, ||_0$. Thus, the constant in (8), $C^{1/d}$, is bounded by $C_0 dp$, for some universal C_0. This proves (5) with $C = C_0$.

To treat the multidimensional case in (5), one may assume that μ is absolutely continuous. As proved in [K-L-S], Theorem 2.7, given non-negative continuous functions f_1, f_2, f_3 and f_4 on $\mathbf{R}^n$, and the numbers $\alpha, \beta > 0$, the following two properties are equivalent:

(a) for every absolutely continuous log-concave probability measure μ on $\mathbf{R}^n$,

$$\left(\int f_1\,d\mu\right)^{\alpha}\left(\int f_2\,d\mu\right)^{\beta} \le \left(\int f_3\,d\mu\right)^{\alpha}\left(\int f_4\,d\mu\right)^{\beta}; \tag{9}$$

(b) for every non-degenerate interval $\Delta \subset \mathbf{R}^n$ with directional vector $v \in \mathbf{R}^n$, and for every $\lambda \in \mathbf{R}$,

$$\left(\int_\Delta e^{\lambda\langle v,x\rangle} f_1(x)\,dx\right)^{\alpha}\left(\int_\Delta e^{\lambda\langle v,x\rangle} f_2(x)\,dx\right)^{\beta} \le \left(\int_\Delta e^{\lambda\langle v,x\rangle} f_3(x)\,dx\right)^{\alpha}\left(\int_\Delta e^{\lambda\langle v,x\rangle} f_4(x)\,dx\right)^{\beta},$$

where dx stands for the Lebesgue measure on Δ.

Thus, (9) reduces to the case where the measure μ is supported by a line and, moreover, represents there the restriction of the exponential measure $e^{\lambda\langle v,x\rangle}\,dx$ to the interval Δ. Applying the above equivalence to $\alpha = 1/p$, $\beta = 1/q$ and to the functions $f_1 = |f|^p$, $f_4 = C^q|f|^q$ $(C > 0)$, and $f_2 = f_3 = 1$, R. Kannan, L. Lovász and M. Simonovits made the following striking conclusion which we state here as a lemma.

Lemma 1. *Let $p > q > 0$ and $C \ge 1$. Given a continuous function f on $\mathbf{R}^n$, the inequality*

$$\|f\|_p \le C\|f\|_q \tag{10}$$

holds true with respect to all log-concave probability measures μ on $\mathbf{R}^n$ if and only if it holds on all intervals in $\mathbf{R}^n$ with respect to the normalized exponential measures.

By continuity, one can clearly consider in (10) the case $q = 0$, as well.

Now, if f is polynomial, its restriction to every line is again a polynomial of the same degree but of one variable. Since the restrictions of the exponential measures are log-concave, the inequality (10) thus reduces to the one-dimensional case.

One may therefore conclude that the inequality (5) holds with $C = C_0$ for all polynomials f on $\mathbf{R}^n$ of degree d in the range $p \ge 1/d$. Applying (5) to $p = k/d$, $k = 1, 2, \dots$, we get

$$\|\,|f|^{1/d}\|_k \le C_0 k\,\|f\|_0^{1/d}.$$

Finally, by Taylor's expansion and using $k^k \le e^k k!$, we obtain that

$$\int \psi_{1/d}\left(\frac{f}{(2eC_0)^d\|f\|_0}\right) d\mu \le 1.$$

Proof of Theorem 2. First, we consider the growth of the constants $C(d,2)$ in (3) in the case $p = 2$. We will now use Lemma 1, with $p = 2$ and $q = 1$, in full volume that gives more than just a reduction of the multidimensional inequality (10) to dimension one. Let Δ be a non-degenerate interval in $\mathbf{R}^n$ with endpoints a, b and with the directional vector $v = (b-a)/|b-a|$. Then, with respect to the normalized exponential measures on Δ, the norms in (10) are given by

$$||f||_p^p = \frac{1}{\int_0^{|b-a|} e^{\lambda x}\, dx} \int_0^{|b-a|} |f(a+xv)|^p e^{\lambda x}\, dx.$$

When f is a polynomial on $\mathbf{R}^n$, $f(a+xv)$ represents a polynomial in $x \in \mathbf{R}$ of the same degree. Moreover, after rescaling, it suffices to consider the case $\lambda = -1$. Therefore, $C(d,2)$ is the optimal constant C in the inequality

$$\left(\int |f|^2\, d\nu_u\right)^{1/2} \leq C \int |f|\, d\nu_u \tag{11}$$

for the class of all polynomials f on $\mathbf{R}$ of degree d with respect to all measures ν_u on $(0,+\infty)$ with densities

$$\frac{d\nu_u(x)}{dx} = \frac{e^{-x}}{1-e^{-u}}\, 1_{(0,u)}(x), \quad u > 0.$$

The limit case represents the exponential measure $\nu_{+\infty} = \nu$ on $(0,+\infty)$ with density e^{-x}, $x > 0$. For a related family of densities, $x^\alpha e^{-x}/\Gamma(\alpha+1)$, the inequality (11), with exponentially increasing constants, was proved by Yu. V. Prokhorov [P2]. He assumed that $\alpha \geq c_0 d$ (for a numerical c_0), but his approach proposed before in [P1] actually works in a more general situation and can be applied in particular to the measures ν_u. Below, to prove (11), we follow Prokhorov's scheme of the proof and simplify his argument about Laguerre's polynomials.

Step 1: $0 < u < 8d$.

Let f be an arbitrary polynomial on $\mathbf{R}$ of degree d (with real coefficients) such that $||f||_2 = 1$ where L^2-norm is understood with respect to ν_u. Let $x_0 \in [0,u]$ be such that $|f(x_0)| = ||f||_\infty = \max_{x\in[0,u]} |f(x)|$, and assume, without loss of generality, that $f(x_0) > 0$. By Taylor's expansion and by Markov's inequality $||f'||_\infty \leq \frac{2d^2}{u}\, ||f||_\infty$, we get, for every point $x \in [0,u]$,

$$\begin{aligned} f(x) &\geq f(x_0) - ||f'||_\infty |x - x_0| \\ &\geq f(x_0) - \frac{2d^2}{u}\, ||f||_\infty |x-x_0| = \left(1 - \frac{2d^2}{u}\, |x-x_0|\right) ||f||_\infty. \end{aligned}$$

Therefore, in the interval $\delta = [x_1, x_2] \equiv [x_0 - u/(4d^2), x_0 + u/(4d^2)] \cap [0,u]$, we have $f(x) \geq \frac{1}{2}\, ||f||_\infty$ so that

$$||f||_1 \geq \int_\delta f(x)\, d\nu_u(x) \geq \frac{1}{2}\, ||f||_\infty\ \nu_u(\delta).$$

In addition, since $x_2 - x_1 \geq u/(4d^2)$, for some middle point $x_3 \in [x_1, x_2]$, we get

$$\nu_u(\delta) = \frac{e^{-x_1} - e^{-x_2}}{1 - e^{-u}} = \frac{x_2 - x_1}{1 - e^{-u}}\, e^{-x_3} \geq \frac{1}{4d^2}\, \frac{u}{1 - e^{-u}}\, e^{-8d} \geq \frac{1}{4d^2}\, e^{-8d}.$$

Hence, $8d^2 e^{8d}\, ||f||_1 \geq ||f||_\infty \geq ||f||_2$ so that (11) is fulfilled with $C = 8d^2 e^{8d}$.

The second step requires some preparation.

Lemma 2. *For every polynomial f on $\mathbf{R}$ of degree $d \geq 1$,*

$$\int_{8d}^{+\infty} |f(x)|^2\, e^{-x}\, dx \leq \frac{1}{2} \int_0^{+\infty} |f(x)|^2\, e^{-x}\, dx.$$

Proof. Assume that $\int_0^{+\infty} |f(x)|^2\, e^{-x}\, dx = ||f||_2^2 = 1$ (with respect to ν) and introduce the Laguerre polynomials

$$L_k(x) = \frac{e^x}{k!}\, \frac{d^k}{dx^k}(x^k e^{-x}) = \sum_{j=0}^{k} (-1)^j C_k^j\, \frac{x^j}{j!}, \quad k = 0, 1, \ldots. \tag{12}$$

They form a complete orthonormal system of functions in $L^2(\nu)$ so that there exists a representation $f = \sum_{k=0}^d a_k L_k$ with $\sum_{k=0}^d |a_k|^2 = 1$. Hence, $|f|^2 \leq \sum_{k=0}^d |L_k|^2$ so that

$$||f||_4^2 = ||\, |f|^2\, ||_2 \leq \sum_{k=0}^{d} ||\, |L_k|^2\, ||_2 = \sum_{k=0}^{d} ||L_k||_4^2. \tag{13}$$

According to (12) and since $(4j)! \leq 4^{4j} j!^4$, we get

$$||L_k||_4 \leq \sum_{j=0}^{k} C_k^j\, \frac{||x^j||_4}{j!} = \sum_{j=0}^{k} C_k^j\, \frac{(4j)!^{\,1/4}}{j!} \leq \sum_{j=0}^{k} C_k^j\, 4^j = 5^k.$$

Thus, by (13), $||f||_4^2 \leq \sum_{k=0}^d 25^k < \frac{25^{d+1}}{24}$. Now, by Cauchy-Schwarz inequality,

$$\int_{8d}^{\infty} |f(x)|^2\, e^{-x}\, dx \leq ||f||_4^2\, ||\, 1_{[8d, +\infty)}\, ||_2 \leq \frac{25^{d+1}}{24}\, e^{-4d} < \frac{1}{2}.$$

Step 2: $u \geq 8d$.

Again, let f be an arbitrary polynomial on $\mathbf{R}$ of degree d such that $||f||_2 = 1$ where L^2-norm is with respect to ν_u. Now, our basic interval will be $[0, 8d]$. Let x_0 be a point of maximum of $|f|$ on this interval, and assume that $f(x_0) > 0$. Again, by Taylor's expansion and by Markov's inequality, for every point $x \in [0, 8d]$,

$$f(x) \geq f(x_0) - ||f'||_\infty |x - x_0| \geq \left(1 - \frac{d}{4}\, |x - x_0|\right) ||f||_\infty$$

(L^∞-norm is taken on $[0, 8d]$). Therefore, in the interval

$$\delta = [x_1, x_2] \equiv [x_0 - 2/d, x_0 + 2/d] \cap [0, 8d]$$

we have $f(x) \geq \frac{1}{2} \|f\|_\infty$ so that

$$\|f\|_1 \geq \int_\delta f(x)\, d\nu_u(x) \geq \frac{1}{2} \|f\|_\infty \ \nu_u(\delta). \tag{14}$$

On the other hand, since $u \geq 8d$ and by Lemma 2,

$$\begin{aligned}\int_0^{8d} |f|^2\, d\nu_u &= \frac{1}{1-e^{-u}} \int_0^{8d} |f(x)|^2 e^{-x}\, dx \geq \frac{1}{2(1-e^{-u})} \int_0^{+\infty} |f(x)|^2 e^{-x}\, dx \\ &\geq \frac{1}{2(1-e^{-u})} \int_0^u |f(x)|^2 e^{-x}\, dx = \frac{1}{2}.\end{aligned}$$

Therefore, $\frac{1}{2} \leq \|f\|_\infty^2 \, \nu_u([0, 8d] \leq \|f\|_\infty^2$. Combining with (14), we get $\|f\|_1 \geq \frac{1}{2\sqrt{2}}\, \nu_u(\delta)$. Using $x_2 - x_1 \geq 2/d$, we obtain that, for some middle point $x_3 \in [x_1, x_2]$,

$$\nu_u(\delta) = \frac{e^{-x_1} - e^{-x_2}}{1-e^{-u}} = \frac{x_2 - x_1}{1 - e^{-u}}\, e^{-x_3} \geq \frac{2}{d}\, e^{-8d}.$$

Hence, $\sqrt{2} d e^{8d} \|f\|_1 \geq \|f\|_2$ so that (11) is fulfilled with $C = \sqrt{2} d\, e^{8d}$. Note that this constant is majorized by the constant $8d^2 e^{8d}$ ($\leq e^{11d}$) obtained on the first step.

Thus, for every polynomial f on $\mathbf{R}^n$ of degree d, with respect to an arbitrary log-concave probability measure μ on $\mathbf{R}^n$,

$$\|f\|_2 \leq e^{11d} \|f\|_1. \tag{15}$$

It remains to consider the general case $p \geq 2$, in order to complete the proof of Theorem 2. One can iterate an inequality of the form (15), $\|f\|_2^2 \leq A^d \|f\|_1^2$, starting from f and successively applying it to the polynomials $f, f^2, f^4, \dots, f^{2^k}$. This yields

$$\|f\|_{2^k}^{2^k} \leq A^{k 2^{k-1} d} \|f\|_1^{2^k}, \quad k \geq 1.$$

Assume $\|f\|_1 = 1$ and pick up $k \geq 1$ such that $2^k \leq p < 2^{k+1}$. Then,

$$\|f\|_p \leq \|f\|_{2^{k+1}} \leq A^{(k+1)d/2} \leq A^{kd} \leq p^{d \log A / \log 2}.$$

According to (15), we can apply these estimates with $A = e^{22}$ and thus get the statement of Theorem 2 with $C = 22/\log 2$.

Acknowledgment

I am grateful to V.D. Milman and Yu.V. Prokhorov for stimulating discussions.

References

[A] Alesker S. (1999) Localization technique on the sphere and the Gromov-Milman theorem on the concentration phenomenon on uniformly convex sphere. Convex Geometric Analysis (Berkeley, CA, 1996), Math. Sci. Res. Inst. Publ., 34, Cambridge Univ. Press, Cambridge

[B] Bobkov S.G. (1998) Isoperimetric and analytic inequalities for log-concave probability measures. Preprint. Ann. Probab., to appear

[B-G] Bobkov S.G., Götze F. (1997) On moments of polynomials. Probab. Theory Appl. 42(3):518–520

[Bor] Borell C. (1974) Convex measures on locally convex spaces. Ark. Math. 12:239–252

[Bou] Bourgain, J. (1991) On the distribution of polynomials on high dimensional convex sets. Lecture Notes in Math. 1469:127–137

[Gu] Guédon O. (1998) Kahane-Khinchine type inequalities for negative exponents. Preprint. Mathematika, to appear

[G-M1] Gromov M., Milman V.D. (1983-84) Brunn theorem and a concentration of volume phenomena for symmetric convex bodies. GAFA Seminar Notes, Tel Aviv University, Israel

[G-M2] Gromov M., Milman V.D. (1987) Generalization of the spherical isoperimetric inequality to uniformly convex Banach spaces. Composition Math. 62:263–282

[K-L-S] Kannan R., Lovász L., Simonovits M. (1995) Isoperimetric problems for convex bodies and a localization lemma. Discrete and Comput. Geom. 13:541–559

[L] Latala R. (1996) On the equivalence between geometric and arithmetic means for logconcave measures. Convex Geometric Analysis, Berkeley, CA, 123–127

[L-S] Lovász L., Simonovits M. (1993) Random walks in a convex body and an improved volume algorithm. Random Structures and Algorithms 4(3):359–412

[M-S] Milman V.D., Schechtman G. (1986) Asymptotic theory of finite dimensional normed spaces. Lecture Notes in Math. 1200, Springer, Berlin

[P1] Prokhorov Yu.V. (1992) On polynomials in normally distributed random variables. Probab. Theory Appl. 37(4):692–694

[P2] Prokhorov Yu.V. (1993) On polynomials in random variables that have the gamma distribution. Probab. Theory Appl. 38(1):198–202