HOMEWORK ASSIGNMENT N4, MATH 4567, SPRING 2019
Due on April 10 (Wednesday)

Problem 1 (7 points). *a*) Given parameters $c > 0$ and $\beta > 0$, show that the Sturm-Liouville boundary value problem

$$\begin{aligned} & y'' + \lambda y = 0, \qquad && y = y(x), \;\; 0 \le x \le c, \\ & y'(0) = \beta\, y(0), && \\ & y'(c) = \beta\, y(c), && \end{aligned}$$

has exactly one negative eigenvalue λ_0 and that this eigenvalue is independent on $c > 0$. Find λ_0 and an associated eigenfunction $y_0(x)$.
b) Determine whether or not $\lambda = 0$ is an eigenvalue. If yes, find an associated eigenfunction.

Problem 2 (6 points). Solve the temperature problem:

$$\begin{aligned} & u_t = k u_{xx}, \qquad && u = u(x,t), \;\; 0 \le x \le \pi, \;\; t > 0 \;\; (k > 0) \\ & u_x(0,t) = \beta\, u(0,t), && \\ & u_x(\pi,t) = \beta\, u(\pi,t), && \\ & u(x,0) = f(x), && \end{aligned}$$

where $f(x)$ is a given continuous function on $[0, \pi]$ and β is a positive parameter. Write your answer in the form of an infinite series

$$u(x,t) = \sum_{n=0}^{\infty} c_n y_n(x) T_n(t). \tag{1}$$

Describe the functions $y_n(x)$ and $T_n(t)$ that are involved and indicate how to compute the coefficients c_n in terms of f.

Problem 3 (6 points). Assume the initial temperatures are constant: let $f(x) = 1$, for definiteness. Determine the first three terms in the representation (1), that is, eventually, evaluate c_0, c_1, and c_2.

Problem 4 (6 points). Given parameters A, B, C (real) and $\beta > 0$, consider the temperature problem with non-homogeneous boundary conditions:

$$\begin{aligned} & u_t = k u_{xx}, \qquad && u = u(x,t), \;\; 0 \le x \le \pi, \;\; t > 0 \;\; (k > 0) \\ & u_x(0,t) = \beta\, u(0,t) + A, && \\ & u_x(\pi,t) = \beta\, u(\pi,t) + B, && \\ & u(x,0) = Cx. && \end{aligned}$$

Reduce it to Problem 2 by virtue of a suitable substitution $u(x,t) = U(x,t) + \Phi(x)$. Indicate new initial temperatures $F(x)$ in the homogeneous problem about $U(x,t)$.