MATH 4567, SPRING 2018
HOMEWORK ASSIGNMENT N5
Due on April 30 (Monday)

Problem 1. Find all eigenvalues and associated eigenfunctions for the Sturm-Liouville problem

$$\begin{aligned} & y''(x) + \lambda y(x) = 0, \qquad 0 \le x \le \pi, \\ & y(0) + y'(0) = 0, \\ & y(\pi) + y'(\pi) = 0. \end{aligned}$$

Problem 2. *a*) Solve the temperature problem in the slab

$$\begin{aligned} & u_t \;=\; k\, u_{xx}, \qquad u = u(x,t),\;\; 0 \le x \le \pi,\;\; t \ge 0 \quad (k > 0) \\ & u_x(0,t) = 0, \\ & u_x(\pi,t) = 0, \\ & u(x,0) \;=\; (1 + \cos^2 x)^2. \end{aligned}$$

b) Find the temperatures in the long run: Show that there is a finite limit $C = \lim_{t\to\infty} u(x,t)$ which is independent of x. Find the constant C.

Note. The initial temperatures may be represented as a cosine polynomial of degree 4.

Problem 3. Use the Fourier transform to solve the temperature problem in the upper half-plane

$$\begin{aligned} & u_t = k u_{xx}, \qquad u = u(x,t),\;\; -\infty < x < \infty,\;\; t \ge 0, \quad u \text{ is bounded}, \\ & u(x,0) = e^{-2x^2}. \end{aligned}$$

Hint. First formulate a general theorem about the boundary value problems

$$\begin{aligned} & u_t \;=\; k\, u_{xx}, \qquad u = u(x,t),\;\; -\infty < x < \infty,\;\; t \ge 0, \quad u \text{ is bounded}, \\ & u(x,0) = f(x). \end{aligned}$$

Problem 4. Use the Fourier transform to solve the boundary value problem, involving the Laplace equation:

$$\begin{aligned} & \Delta u = 0, \qquad u = u(x,y),\;\; x, y \ge 0, \quad u \text{ is bounded}, \\ & u_x(0,y) = 0, \\ & u(x,0) = \tfrac{1}{1+x^2}. \end{aligned}$$

Hint. First formulate a general theorem about boundary value problems of the form

$$\begin{aligned} & \Delta u = 0, \qquad u = u(x,y),\;\; x, y \ge 0, \quad u \text{ is bounded}, \\ & u_x(0,y) = 0, \\ & u(x,0) = f(x). \end{aligned}$$

Note that the Fourier transform for the functions $f(x) = e^{-x^2/(2\sigma^2)}$ and $f(x) = \frac{1}{1+x^2}$ are known and have been evaluated in class.