HOMEWORK ASSIGNMENT N4, MATH 4567, SPRING 2018
Due on April 16 (Monday) – problems and solutions

Problem 1. *a*) Given parameters $c > 0$ and $\beta > 0$, show that the Sturm-Liouville boundary value problem

$$\begin{aligned} &y'' + \lambda y = 0, \qquad y = y(x),\ 0 \le x \le c,\\ &y'(0) = \beta\, y(0),\\ &y'(c) = \beta\, y(c), \end{aligned}$$

has exactly one negative eigenvalue λ_0 and that this eigenvalue is independent on $c > 0$. Find λ_0 and an associated eigenfunction $y_0(x)$.

b) Determine whether or not $\lambda = 0$ is an eigenvalue. If yes, find an associated eigenfunction.

Solution. We know that any Sturm-Liouville problem

$$\begin{aligned} &y''(x) + \lambda y(x) = 0, \qquad 0 \le x \le c,\\ &\alpha_1 y(0) + \alpha_2 y'(0) = 0,\\ &\beta_1 y(c) + \beta_2 y'(c) = 0 \end{aligned}$$

has only non-negative eigenvalues λ, when $\alpha_1\alpha_2 \le 0$ and $\beta_1\beta_2 \ge 0$. In our case, $\alpha_1 = 1$, $\alpha_2 = -\beta$, $\beta_1 = 1$, $\beta_2 = -\beta$, so this condition is not fulfilled, and one should consider separately the three cases to find all eigenvalues. Actually, in this problem, we are asked about non-positive λ.

a) Case 1: $\lambda = -\alpha^2$ $(\alpha > 0)$. The general solution to the Sturm-Liouville equation $y''(x) + \lambda y(x) = 0$ may be described in this case by the formula

$$y = C_1 \cosh(\alpha x) + C_2 \sinh(\alpha x)$$

with arbitrary coefficients C_1 and C_2, where we use the hyperbolic functions $\cosh(t) = \frac{e^t + e^{-t}}{2}$ and $\sinh(t) = \frac{e^t - e^{-t}}{2}$. Hence

$$y(0) = C_1, \qquad y' = \alpha C_1 \sinh(\alpha x) + \alpha C_2 \cosh(\alpha x), \quad y'(0) = \alpha C_2,$$

and the first boundary condition becomes

$$\alpha C_2 = \beta C_1.$$

This gives a necessary relationship between the coefficients. Since we are looking for a non-trivial (not identically zero) solution $y = y(x)$ up to a multiplicative factor, one may take any fixed value of $C_1 \ne 0$, and then the unique choice for the other coefficient will be $C_2 = \beta C_1/\alpha$. To work with simpler expressions, let us choose $C_1 = \alpha$, so that $C_2 = \beta$. The solution $y(x)$ and its derivative $y'(x)$ are thus simplified to

$$y = \alpha \cosh(\alpha x) + \beta \sinh(\alpha x), \qquad y' = \alpha^2 \sinh(\alpha x) + \beta\alpha \cosh(\alpha x).$$

In particular,

$$y(c) = \alpha \cosh(\alpha c) + \beta \sinh(\alpha c), \qquad y'(c) = \alpha^2 \sinh(\alpha c) + \beta\alpha \cosh(\alpha c),$$

and the second boundary condition becomes

$$\alpha^2 \sinh(\alpha c) + \beta\alpha \cosh(\alpha c) = \beta\alpha \cosh(\alpha c) + \beta^2 \sinh(\alpha c).$$

Since $\sinh(\alpha c) > 0$, this is equivalent to $\alpha^2 = \beta^2$, which is uniquely solved as $\alpha = \beta$ (in positive numbers). Therefore,

$$\lambda = -\alpha^2 = -\beta^2, \qquad y = \beta \cosh(\beta x) + \beta \sinh(\beta x) = \beta\, e^{\beta x}.$$

Since we are looking for a non-trivial solution $y = y(x)$ up to a multiplicative factor, the constant β may be removed.

b) Case 2: $\lambda = 0$. Then the general solution is $y = C_1 + C_2 x$, and the boundary conditions become

$C_2 = \beta C_1,$
$C_2 = \beta\,(C_1 + C_2 c).$

This linear system is solved as $C_1 = C_2 = 0$, which means that this case is impossible.

Answer.

a) $\lambda_0 = -\beta^2$ is the unique negative eigenvalue with eigenfunction $y_0 = e^{\beta x}$.
b) $\lambda = 0$ is not an eigenvalue of the given Sturm-Liouville problem.

Problem 2. Solve the temperature problem:

$$\begin{aligned} &u_t = k u_{xx}, && u = u(x,t),\ \ 0 \le x \le \pi,\ \ t > 0\ \ (k > 0) \\ &u_x(0,t) = \beta\, u(0,t), \\ &u_x(\pi,t) = \beta\, u(\pi,t), \\ &u(x,0) = f(x), \end{aligned}$$

where $f(x)$ is a given continuous function on $[0,\pi]$ and β is a positive parameter. Write your answer in the form of an infinite series

$$u(x,t) = \sum_{n=0}^{\infty} c_n y_n(x) T_n(t). \tag{1}$$

Describe the functions $y_n(x)$ and $T_n(t)$ that are involved and indicate how to compute the coefficients c_n in terms of f.

Solution. As we know, the solution to the given temperature problem is described by the functional series (1) in which y_n represent the eigenfunctions of the associated Sturm-Liouville boundary value problem, namely

$$
\begin{aligned}
&y'' + \lambda y = 0, \qquad\qquad y = y(x),\ \ 0 \le x \le \pi,\\
&y'(0) = \beta\, y(0),\\
&y'(\pi) = \beta\, y(\pi),
\end{aligned}
$$

and $T_n(t) = e^{-\lambda_n kt}$, where λ_n is the eigenvalue to which the eigenfunction y_n belongs. Moreover, the coefficients c_n in (1) are determined by the last non-homogeneous boundary condition via the orthogonal series expansion

$$
f(x) = \sum_{n=0}^{\infty} c_n y_n(x).
$$

Here, necessarily

$$
c_n = \frac{\langle f, y_n\rangle}{\|y_n\|^2}, \qquad \langle f, y_n\rangle = \int_0^\pi f(x) y_n(x)\, dx, \quad \|y_n\|^2 = \int_0^\pi y_n(x)^2\, dx.
$$

As one can notice, the associated Sturm-Liouville boundary value problem is exactly the same as the one in Problem 1 with $c = \pi$. So, we already know that this problem has a unique negative eigenvalue $\lambda_0 = -\beta^2$ with its eigenfunction $y_0(x) = e^{\beta x}$. Hence, $T_0(t) = e^{\beta^2 kt}$. We also know that all remaining eigenvalues are positive. To find these eigenvalues and the eigenfunctions, we need to consider:

Case 3: $\lambda = \alpha^2$ ($\alpha > 0$). In this case, the general solution to the Sturm-Liouville equation $y''(x) + \lambda y(x) = 0$ is given by

$$
y = C_1 \cos(\alpha x) + C_2 \sin(\alpha x), \qquad y(0) = C_1,
$$

for which we have

$$
y' = \alpha C_2 \cos(\alpha x) - \alpha C_1 \sin(\alpha x), \qquad y'(0) = \alpha C_2.
$$

Hence $\alpha C_2 = \beta C_1$, by the first boundary condition. Without loss of generality, put $C_1 = \alpha$, $C_2 = \beta$, so that

$$
y = \alpha \cos(\alpha x) + \beta \sin(\alpha x), \qquad y' = \beta\alpha \cos(\alpha x) - \alpha^2 \sin(\alpha x).
$$

Involving the second boundary condition, we obtain that α should solve the equation

$$
\beta\alpha \cos(\alpha\pi) - \alpha^2 \sin(\alpha\pi) = \beta\alpha \cos(\alpha\pi) + \beta^2 \sin(\alpha\pi)
$$

which is equivalent to $\sin(\alpha\pi) = 0$. It is solved as $\alpha = \alpha_n = n$, $n \ge 1$, in which case

$$
y = y_n(x) = n\ \cos(nx) + \beta \sin(nx).
$$

Let us also specify the L^2-norms of the eigenfunctions: For $n = 0$ we have

$$
\|y_0\|^2 = \int_0^\pi y_0(x)^2\, dx = \int_0^\pi e^{2\beta x}\, dx = \frac{e^{2\beta\pi} - 1}{2\beta},
$$

and for integers $n \geq 1$,

$$
\begin{aligned}
\|y_n\|^2 &= \int_0^\pi (n\,\cos(nx) + \beta\sin(nx))^2\,dx \\
&= n^2 \int_0^\pi \cos^2(nx)\,dx + 2\beta n \int_0^\pi \cos(nx)\sin(nx)\,dx + \beta^2 \int_0^\pi \sin^2(nx)\,dx \\
&= n^2 \int_0^\pi \frac{1+\cos(2nx)}{2}\,dx + \beta n \int_0^\pi \sin(2nx)\,dx + \beta^2 \int_0^\pi \frac{1-\cos(2nx)}{2}\,dx \\
&= \frac{\pi}{2}\,(n^2+\beta^2).
\end{aligned}
$$

Here we used the obvious identities

$$\int_0^\pi \cos(2nx)\,dx = \int_0^\pi \sin(2nx)\,dx = 0.$$

Answer. The solution to the temperature problem is given by the functional series (1), where

$$
\begin{aligned}
y_0(x) &= e^{\beta x}, \qquad & y_n(x) &= n\,\cos(nx) + \beta\sin(nx) \quad (n \geq 1), \\
T_0(t) &= e^{\beta^2 kt}, \qquad & T_n(t) &= e^{-nkt} \quad (n \geq 1),
\end{aligned}
$$

with coefficients

$$
\begin{aligned}
c_0 &= \frac{2\beta}{e^{2\beta\pi}-1} \int_0^\pi f(x)\,e^{\beta x}\,dx, \\
c_n &= \frac{2}{\pi\,(n^2+\beta^2)} \int_0^\pi f(x)\,(n\,\cos(nx) + \beta\sin(nx))\,dx \quad (n \geq 1).
\end{aligned}
$$

Problem 3. Assume that the initial temperatures are constant: Let $f(x) = 1$. Determine the first three terms in the representation (1), that is, eventually, evaluate c_0, c_1, and c_2.

Solution. Using the answer to Problem 2, we find that

$$
\begin{aligned}
c_0 &= \frac{2\beta}{e^{2\beta\pi}-1} \int_0^\pi e^{\beta x}\,dx \;=\; 2\,\frac{e^{\beta\pi}-1}{e^{2\beta\pi}-1}, \\
c_1 &= \frac{2}{\pi\,(1+\beta^2)} \int_0^\pi (\cos x + \beta\sin x)\,dx \;=\; \frac{4\beta}{\pi\,(1+\beta^2)}, \\
c_2 &= \frac{2}{\pi\,(4+\beta^2)} \int_0^\pi (2\cos(2x) + \beta\sin(2x))\,dx \;=\; 0.
\end{aligned}
$$

Answer. The solution to the temperature problem with the first three terms is given by

$$u(x,t) \;=\; 2\,\frac{e^{\beta\pi}-1}{e^{2\beta\pi}-1}\,e^{\beta x}\,e^{\beta^2 kt} + \frac{4\beta}{\pi\,(1+\beta^2)}\,(\cos x + \beta\sin x)\,e^{-kt} + 0 + \ldots$$

Problem 4. Given parameters A, B, C and $\beta > 0$, consider the temperature problem with non-homogeneous boundary conditions:

$$\begin{aligned}
&u_t = k u_{xx}, && u = u(x,t),\ \ 0 \le x \le \pi,\ \ t > 0\ \ (k > 0)\\
&u_x(0,t) = \beta\, u(0,t) + A,\\
&u_x(\pi,t) = \beta\, u(\pi,t) + B,\\
&u(x,0) = Cx.
\end{aligned}$$

Reduce it to Problem 2 by virtue of a suitable substitution $u(x,t) = U(x,t) + \Phi(x)$. Indicate new initial temperatures $F(x)$ in the homogeneous problem about $U(x,t)$.

Solution. One may look for a suitable substitution $u(x,t) = U(x,t) + \Phi(x)$ with a linear function $\Phi(x) = C_1 + C_2\, x$, in which case the problem becomes

$$\begin{aligned}
&U_t = k U_{xx}, && U = U(x,t),\ \ 0 \le x \le \pi,\ \ t > 0\ \ (k > 0)\\
&U_x(0,t) + C_2 = \beta\,(U(0,t) + C_1) + A,\\
&U_x(\pi,t) + C_2 = \beta\,(U(\pi,t) + C_1 + C_2\pi) + B,\\
&U(x,0) + (C_1 + C_2\, x) = Cx.
\end{aligned}$$

In order to reach homogeneous vertical boundary conditions, the unknown parameters should satisfy

$$\begin{aligned}
&C_2 = \beta\, C_1 + A,\\
&C_2 = \beta\,(C_1 + C_2\pi) + B.
\end{aligned}$$

This is a linear system in two unknowns which is easily solved. Equalizing the two equalities, we get $A = C_2\, \beta\pi + B$, so

$$C_2 = \frac{A - B}{\beta\pi}.$$

Inserting this into the second equality, we also find

$$C_1 = \frac{1 - \beta\pi}{\beta^2\pi}\, A - \frac{1}{\beta^2\pi}\, B.$$

Answer. The new problem with homogeneous vertical boundary conditions is

$$\begin{aligned}
&U_t = k U_{xx}, && U = U(x,t),\ \ 0 \le x \le \pi,\ \ t > 0\ \ (k > 0)\\
&U_x(0,t) = \beta\, U(0,t),\\
&U_x(\pi,t) = \beta\, U(\pi,t),\\
&U(x,0) = F(x)
\end{aligned}$$

with new initial temperatures

$$F(x) = Cx - \Big(\frac{1 - \beta\pi}{\beta^2\pi}\, A - \frac{1}{\beta^2\pi}\, B\Big) - \frac{A - B}{\beta\pi}\, x.$$