HOMEWORK ASSIGNMENT N4, MATH 4567, SPRING 2015
Due on April 24 (Friday)

This assignment is a continuation of the problem from the previous homework assignment: Given parameters $c > 0$ and $\beta > 0$, show that the Sturm-Liouville boundary value problem

$$\begin{aligned} &y'' + \lambda y = 0, \qquad y = y(x), \ \ 0 \le x \le c, \\ &y'(0) = \beta\, y(0), \\ &y'(c) = \beta\, y(c), \end{aligned}$$

has exactly one negative eigenvalue λ_0 and that this eigenvalue is independent on c. Find λ_0 and an associated eigenfunction y_0. Determine whether $\lambda = 0$ is an eigenvalue. If yes, find an associated eigenfunction.

Problem 1. Solve the temperature problem:

$$\begin{aligned} &u_t = k u_{xx}, \qquad u = u(x,t), \ \ 0 \le x \le \pi, \ \ t > 0 \ \ (k > 0) \\ &u_x(0,t) = \beta\, u(0,t), \\ &u_x(\pi,t) = \beta\, u(\pi,t), \\ &u(x,0) = f(x), \end{aligned}$$

where $f(x)$ is a given continuous function on $[0, \pi]$ and β is a positive parameter. Write your answer in the form of an infinite series

$$u(x,t) = \sum_{n=0}^{\infty} c_n y_n(x) T_n(t). \tag{1}$$

Describe the functions $y_n(x)$ and $T_n(t)$ that are involved and indicate how to compute the coefficients c_n in terms of f.

Problem 2. Assume the initial temperatures are constant: let $f(x) = 1$, for definiteness. Determine the first three terms in the representation (1), that is, eventually, evaluate c_0, c_1, and c_2.

Problem 3. Given parameters A, B, C (real) and $\beta > 0$, consider the temperature problem with non-homogeneous boundary conditions:

$$\begin{aligned} &u_t = k u_{xx}, \qquad u = u(x,t), \ \ 0 \le x \le \pi, \ \ t > 0 \ \ (k > 0) \\ &u_x(0,t) = \beta\, u(0,t) + A, \\ &u_x(\pi,t) = \beta\, u(\pi,t) + B, \\ &u(x,0) = Cx. \end{aligned}$$

Reduce it to Problem 1 by virtue of a suitable substitution $u(x,t) = U(x,t) + \Phi(x)$. Indicate new initial temperatures $F(x)$ in the homogeneous problem about $U(x,t)$.

Problem 4. *a*) Describe the solution $u(x,t)$ to the above non-homogeneous problem in the form similar to (1). Do not evaluate inner products and L^2-norms (but write formulas for them using definite integrals).

b) Determine the relationship between the parameters A, B, C, under which the solution $u(x,t)$ is constant in time (that is, depends on x, only).