

HOMEWORK ASSIGNMENT N4, MATH 4567, SPRING 2014
Due on April 21 (Monday)

This assignment is a continuation of the problem from the previous homework assignment: Given parameters $c > 0$ and $\beta > 0$, show that the Sturm-Liouville boundary value problem

$$\begin{aligned}y'' + \lambda y &= 0, & y &= y(x), \quad 0 \leq x \leq c, \\y'(0) &= \beta y(0), \\y'(c) &= \beta y(c),\end{aligned}$$

has exactly one negative eigenvalue λ_0 and that this eigenvalue is independent on c . Find λ_0 and an associated eigenfunction y_0 . Determine whether $\lambda = 0$ is an eigenvalue. If yes, find an associated eigenfunction.

Problem 1. Solve the temperature problem:

$$\begin{aligned}u_t &= k u_{xx}, & u &= u(x, t), \quad 0 \leq x \leq \pi, \quad t > 0 \quad (k > 0) \\u_x(0, t) &= \beta u(0, t), \\u_x(\pi, t) &= \beta u(\pi, t), \\u(x, 0) &= f(x),\end{aligned}$$

where $f(x)$ is a given continuous function on $[0, \pi]$ and β is a positive parameter. Write your answer in the form of an infinite series

$$u(x, t) = \sum_{n=0}^{\infty} c_n y_n(x) T_n(t). \quad (1)$$

Describe the functions $y_n(x)$ and $T_n(t)$ that are involved and indicate how to compute the coefficients c_n in terms of f .

Problem 2. Assume the initial temperatures are constant: let $f(x) = 1$, for definiteness. Determine the first three terms in the representation (1), that is, eventually, evaluate c_0 , c_1 , and c_2 .

Problem 3. Given parameters A, B, C (real) and $\beta > 0$, consider the temperature problem with non-homogeneous boundary conditions:

$$\begin{aligned}u_t &= k u_{xx}, & u &= u(x, t), \quad 0 \leq x \leq \pi, \quad t > 0 \quad (k > 0) \\u_x(0, t) &= \beta u(0, t) + A, \\u_x(\pi, t) &= \beta u(\pi, t) + B, \\u(x, 0) &= Cx.\end{aligned}$$

Reduce it to Problem 1 by virtue of a suitable substitution $u(x, t) = U(x, t) + \Phi(x)$. Indicate new initial temperatures $F(x)$ in the homogeneous problem about $U(x, t)$.

Problem 4. a) Describe the solution $u(x, t)$ to the above non-homogeneous problem in the form similar to (1). Do not evaluate inner products and L^2 -norms (but write formulas for them using definite integrals).

b) Determine the relationship between the parameters A, B, C , under which the solution $u(x, t)$ is constant in time (that is, depends on x , only).