Worksheet 3

Topics: partial derivatives, tangent plane of a surface.

1.

a) Let $u(x,y)=\ln(e^x+e^y)$. Compute $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}$.

Solution.

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{1}{e^x+e^y}\cdot\frac{\partial}{\partial x}\left(e^x+e^y\right)=\frac{e^x}{e^x+e^y}\\ \frac{\partial u}{\partial y}&=\frac{1}{e^x+e^y}\cdot\frac{\partial}{\partial y}\left(e^x+e^y\right)=\frac{e^y}{e^x+e^y}\\ \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}&=\frac{e^x}{e^x+e^y}+\frac{e^y}{e^x+e^y}=1.\end{aligned}$$

b) Let $u(x,y,z)=(x-y)(y-z)(z-x)$. Compute $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$.

Solution.

$$\begin{aligned}\frac{\partial u}{\partial x}=(y-z)\cdot\frac{\partial u}{\partial x}\left[(x-y)(z-x)\right]&=(y-z)\cdot\left[1\cdot(z-x)+(x-y)\cdot(-1)\right]\\ &=(y-z)(z-x)-(x-y)(y-z)\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial y}=(z-x)\cdot\frac{\partial u}{\partial y}\left[(x-y)(y-z)\right]&=(z-x)\cdot\left[(-1)\cdot(y-z)+(x-y)\cdot 1\right]\\ &=-(y-z)(z-x)+(x-y)(z-x)\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial z}=(x-y)\cdot\frac{\partial u}{\partial z}\left[(y-z)(z-x)\right]&=(x-y)\cdot\left[(-1)\cdot(z-x)+(y-z)\cdot 1\right]\\ &=-(x-y)(z-x)+(x-y)(y-z)\end{aligned}$$

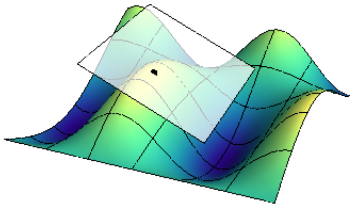
$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$$

2. Let $u(x,y,z)=x^{y^z}$. Find all first-order partial derivatives of u.

Solution.

$$\begin{aligned}\frac{\partial u}{\partial x}&=x^{(y^z-1)}\cdot y^z\\ \frac{\partial u}{\partial y}&=x^{y^z}\cdot\ln(x)\cdot\frac{\partial}{\partial y}\left(y^z\right)=x^{y^z}\cdot\ln(x)\cdot z\cdot y^{z-1}\\ \frac{\partial u}{\partial z}&=x^{y^z}\cdot\ln(x)\cdot\frac{\partial}{\partial z}\left(y^z\right)=x^{y^z}\cdot\ln(x)\cdot y^z\cdot\ln(y)\end{aligned}$$

Consider a surface S defined by the equation $z = f(x, y)$. The equation of the *plane* tangent to S at the point $\big(x_0, y_0, f(x_0, y_0)\big)$ is



$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} \cdot (x - x_0) + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} \cdot (y - y_0),$$

provided that $f = f(x, y)$ is differentiable at $x = x_0$, $y = y_0$.

3. a) Find an equation of the plane tangent to the surface $z = \frac{x^2}{2} - xy + 1$ at the point $(2, 1, 1)$.

Solution. If $f(x, y) = \frac{x^2}{2} - xy + 1$, then

$$\begin{aligned} f(2,1) &= 1 \\ \frac{\partial f}{\partial x}\Big|_{(2,1)} &= \left(\frac{2x}{2} - y\right)\Big|_{(2,1)} = 1 \\ \frac{\partial f}{\partial y}\Big|_{(2,1)} &= (-x)\Big|_{(2,1)} = -2. \end{aligned}$$

So the tangent plane at the point $(2, 1, 1)$ has an equation

$$z = 1 + 1 \cdot (x - 2) + (-2) \cdot (y - 1),$$

or, after simplification, $x - 2y - z + 1 = 0$.

b) Find an equation of the plane tangent to the surface $x + 2y - \ln z + 4 = 0$ at the point $(2, -3, 1)$.

Solution. First, we need to express z as a function of x and y from the given equation. We have

$$\begin{aligned} \ln z &= x + 2y + 4 \\ z &= e^{x+2y+4}. \end{aligned}$$

If we take $f(x, y) = e^{x+2y+4}$, then

$$\begin{aligned} f(2,-3) &= e^{2-6+4} = 1 \\ \frac{\partial f}{\partial x}\Big|_{(2,-3)} &= \left(e^{x+2y+4}\right)\Big|_{(2,-3)} = 1 \\ \frac{\partial f}{\partial y}\Big|_{(2,-3)} &= \left(2e^{x+2y+4}\right)\Big|_{(2,-3)} = 2. \end{aligned}$$

Then the tangent plane at the point $(2, -3, 1)$ has an equation

$$z = 1 + 1 \cdot (x - 2) + 2 \cdot (y - (-3)).$$

After simplification, it becomes

$$x + 2y - z + 5 = 0.$$

3. Find all points on the surface $z = x^2y + 3xy + 6x + 1$ where the tangent plane is

a) horizontal; b) parallel to the plane $-3x + 2y + z = 10$.

Solution. Recall that the plane given by an equation $Ax + By + Cz + D = 0$ has the vector $\mathbf{n} = (A, B, C)$ as its normal vector. Hence, in particular, the vector

$$\mathbf{n} = \left(\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}, \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}, -1 \right) \tag{1}$$

is a normal vector of the tangent plane

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} \cdot (x - x_0) + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} \cdot (y - y_0)$$

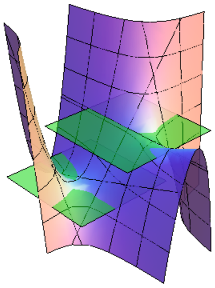
a) A horizontal plane has a vertical normal vector. The x- and y-components of such a vector are both zero. Hence, according to (1), to find a horizontal tangent plane, we need to solve the system of equations

$$\begin{cases} \frac{\partial f}{\partial x} & = 0 \\ \frac{\partial f}{\partial y} & = 0 \end{cases}$$

In our case, $f(x, y) = x^2y + 3xy + 6x + 1$ and we obtain

$$\begin{cases} \frac{\partial f}{\partial x} & = 2xy + 3y + 6 = 0 \\ \frac{\partial f}{\partial y} & = x^2 + 3x = 0 \end{cases}$$

The second equation implies $x = 0$ or $x = -3$. Then the first equation yields $y = -2$ or $y = 2$ respectively. So the planes tangent to the given surface are horizontal at the points $(0, -2, f(0, -2)) = (0, -2, 1)$ and $(-3, 2, f(-3, 2)) = (-3, 2, -17)$.



b) Two distinct planes are parallel if and only if their normal vectors are parallel, and two vectors are parallel if and only if one of them is a multiple of the other. Hence, to find a tangent plane parallel to the plane $-3x + 2y + z = 10$, we need to find the points where the vector

$$\mathbf{n} = \left(\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}, \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}, -1 \right)$$

is a multiple of $\mathbf{m} = (-3, 2, 1)$. Comparing the z-components, we conclude that $\mathbf{n}$ must be equal to $(-1)\mathbf{m}$. It gives us the system of equations

$$\begin{cases} \frac{\partial f}{\partial x} = 2xy + 3y + 6 = 3 \\ \frac{\partial f}{\partial x} = x^2 + 3x = -2 \end{cases}$$

Solving the second equation, we obtain $x = -2$ or $x = -1$. Then the first one yields $y = 3$ or $y = -3$ respectively. Thus the planes tangent to the given surface at the points $(-2, 3, f(-2, 3)) = (-2, 3, -17)$ and $(-1, -3, f(-1, -3)) = (-1, -3, 1)$ are parallel to $-3x + 2y + z = 10$.