

Worksheet 3 $\frac{1}{2}$ 

Topics: tangent planes, linear approximations.

1. Show that the surface given by the equation  $3x^2 + 3y^2 - 6xy - z = 10x$  has no horizontal tangent planes.

**Solution.** Let  $f(x, y) = 3x^2 + 3y^2 - 6xy - 10x$ . Then the plane tangent to the surface  $z = f(x, y)$  at some point  $(x_0, y_0, f(x_0, y_0))$  has an equation

$$z = \left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} \cdot (x - x_0) + \left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)} \cdot (y - y_0) + f(x_0, y_0)$$

that is,

$$z = (6x_0 - 6y_0 - 10) \cdot (x - x_0) + (6y_0 - 6x_0) \cdot (y - y_0) + f(x_0, y_0).$$

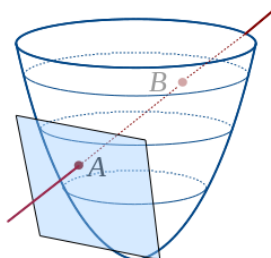
A plane defined by the equation  $z = Ax + By + C$  is horizontal exactly when  $A = B = 0$ . In our case,  $A = 6x_0 - 6y_0 - 10$ ,  $B = 6y_0 - 6x_0$ , and we can see that  $A$  and  $B$  are never equal to zero simultaneously. Hence, there are no horizontal planes tangent to the given surface.

2. The inner surface of a teacup is a paraboloid  $z = x^2 + y^2 - 8$ . A wasp, sitting in the cup at the point  $(3, 0, 1)$ , takes off perpendicular to the cup's surface and flies straight until it reaches the opposite point. How far does the wasp fly?



**Solution.** The problem can be restated in geometric terms as follows:

Given a paraboloid  $z = x^2 + y^2 - 8$  and a line passing through the point  $A(3, 0, 1)$  perpendicular to the paraboloid. Let  $B$  be the other point where the line intersects the surface. Find the distance  $|AB|$ .



**Step 1.** We find a vector orthogonal to the given surface at the point  $A$ .

The vector

$$\mathbf{n} = \left( \frac{\partial f}{\partial x} \Big|_{(x_0, y_0)}, \frac{\partial f}{\partial y} \Big|_{(x_0, y_0)}, -1 \right)$$

is a normal vector of the plane tangent to the surface  $z = f(x, y)$  at the point  $(x_0, y_0, z_0)$ . We compute the partial derivatives of the function  $f(x, y) = x^2 + y^2 - 8$  at the point  $A(3, 0, 1)$ :

$$\begin{aligned} \frac{\partial f}{\partial x} \Big|_{(3,0)} &= 2x = 6 \\ \frac{\partial f}{\partial y} \Big|_{(3,0)} &= 2y = 0 \end{aligned}$$

Then  $\mathbf{n} = (6, 0, -1)$  is a normal vector of the tangent plane. This vector is orthogonal to the paraboloid at the point  $A$ .

**Step 2.** We find an equation of the line passing through the point  $(3, 0, 1)$  in the direction given by the vector  $\mathbf{n} = (6, 0, -1)$ .

An equation of the line passing through the point  $(x_0, y_0, z_0)$  in the direction of the vector  $\mathbf{d} = (a, b, c)$  is

$$(x, y, z) = (x_0, y_0, z_0) + (a, b, c)t.$$

In our case  $(x_0, y_0, z_0) = (3, 0, 1)$  and the direction is given by the vector  $\mathbf{n} = (6, 0, -1)$ . Hence, an equation of the line is

$$(x, y, z) = (3, 0, 1) + (6, 0, -1)t,$$

that is,

$$\begin{cases} x = 3 + 6t \\ y = 0 \\ z = 1 - t \end{cases}.$$

**Step 3.** We find the coordinates of the point of intersection of the line and the paraboloid  $z = x^2 + y^2 - 8$  other than  $A(3, 0, 1)$ .

To find the points of intersection of the line and the paraboloid, we need to solve the system of equations

$$\begin{cases} z = x^2 + y^2 - 8 \\ x = 3 + 6t \\ y = 0 \\ z = 1 - t \end{cases}$$

Plugging the expressions for  $x$ ,  $y$  and  $z$  given in the last three equations into the first equation, we obtain

$$\begin{aligned} (1 - t) &= (3 + 6t)^2 + 0^2 - 8 \\ 36t^2 + 37t &= 0. \end{aligned}$$

Thus,  $t = 0$  or  $t = -\frac{37}{36}$ , which implies

$$x = 3, \quad y = 0, \quad z = 1$$

or

$$x = 3 + 6 \cdot \left(-\frac{37}{36}\right), \quad y = 0, \quad z = 1 - \left(-\frac{37}{36}\right).$$

The first solution is exactly point  $A$  and the second solution gives us another point  $B$ .

**Step 4.** We find the distance  $|AB|$ .

$$|AB| = \sqrt{6^2 \cdot \left(-\frac{37}{36}\right)^2 + 0^2 + (-1)^2 \cdot \left(-\frac{37}{36}\right)^2} = \frac{37\sqrt{37}}{36}.$$

The values of the *linear* function

$$z(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}\bigg|_{(x_0, y_0)} \cdot (x - x_0) + \frac{\partial f}{\partial y}\bigg|_{(x_0, y_0)} \cdot (y - y_0)$$

are close to the actual values of a differentiable function  $f = f(x, y)$  when  $x$  and  $y$  are close to  $x_0, y_0$ . This property can be used to perform quick approximate computations.

3. Use the linear approximation formula to estimate

a)  $\ln(\sqrt{1.04} + \sqrt[3]{1.01} - 1)$ ; b)  $1.04^{2.02}$ .

**Solution.**

a) We let  $f(x, y) = \ln(\sqrt{x} + \sqrt[3]{y} - 1)$ ,  $x_0 = 1$ ,  $y_0 = 1$  and compute

$$\begin{aligned} f(x_0, y_0) &= \ln(\sqrt{x} + \sqrt[3]{y} - 1)\bigg|_{(1,1)} = 0 \\ \frac{\partial f}{\partial x}\bigg|_{(x_0, y_0)} &= \frac{1}{\sqrt{x} + \sqrt[3]{y} - 1} \cdot \frac{1}{2\sqrt{x}}\bigg|_{(1,1)} = \frac{1}{2} \\ \frac{\partial f}{\partial y}\bigg|_{(x_0, y_0)} &= \frac{1}{\sqrt{x} + \sqrt[3]{y} - 1} \cdot \frac{1}{3\sqrt[3]{y^2}}\bigg|_{(1,1)} = \frac{1}{3} \end{aligned}$$

Then the linear approximation for  $f$  at the point  $(1, 1)$  is

$$z(x, y) = 0 + \frac{1}{2} \cdot (x - 1) + \frac{1}{3} \cdot (y - 1)$$

and we obtain

$$\ln(\sqrt{1.04} + \sqrt[3]{1.01} - 1) \approx z(1.04, 1.01) = 0 + \frac{1}{2} \cdot (1.04 - 1) + \frac{1}{3} \cdot (1.01 - 1) = 0.0233\dots$$

b) We let  $f(x, y) = x^y$ ,  $x_0 = 1$ ,  $y_0 = 2$  and compute

$$\begin{aligned} f(x_0, y_0) &= x^y\bigg|_{(1,2)} = 1 \\ \frac{\partial f}{\partial x}\bigg|_{(x_0, y_0)} &= yx^{y-1}\bigg|_{(1,2)} = 2 \\ \frac{\partial f}{\partial y}\bigg|_{(x_0, y_0)} &= x^y \ln x\bigg|_{(1,2)} = 0 \end{aligned}$$

Then the linear approximation for  $f$  at the point  $(1, 2)$  is

$$z(x, y) = 1 + 2 \cdot (x - 1) + 0 \cdot (y - 2)$$

and we obtain

$$1.04^{2.02} \approx z(1.04, 2.02) = 1 + 2 \cdot (1.04 - 1) + 0 \cdot (2.02 - 2) = 1.08.$$

4. Using the linear approximation formula, decide what makes a greater impact on the value of  $z(x, y) = x^y$  at  $x = y = 10$ : increasing  $x$  by 0.2 or increasing  $y$  by 0.1.

**Solution.** We compute

$$\begin{aligned} f(10, 10) &= 10^{10} \\ \left. \frac{\partial f}{\partial x} \right|_{(10,10)} &= yx^{y-1} \Big|_{(10,10)} = 10^{10} \\ \left. \frac{\partial f}{\partial y} \right|_{(10,10)} &= x^y \ln x \Big|_{(10,10)} = 10^{10} \ln 10. \end{aligned}$$

So the linear approximation for  $f$  at the point  $(10, 10)$  is

$$z(x, y) = 10^{10} + 10^{10} \cdot (x - 10) + 10^{10} \ln 10 \cdot (y - 10).$$

Then we have

$$\begin{aligned} 10.2^{10} - 10^{10} &\approx z(10.2, 10) - 10^{10} = 10^{10} \cdot 0.2 \\ 10^{10.1} - 10^{10} &\approx z(10, 10.1) - 10^{10} = 10^{10} \ln 10 \cdot 0.1 \end{aligned}$$

It remains to compare  $0.2$  and  $\ln 10 \cdot 0.1$ . Since  $e = 2.71828\dots$ , we have  $e^2 < 9 < 10 = e^{\ln 10}$ . Since  $f(t) = e^t$  is a strictly increasing function, then the last inequality implies  $2 < \ln 10$  and  $0.2 < \ln 10 \cdot 0.1$ . Therefore, increasing  $y$  by 0.1 makes a greater impact.