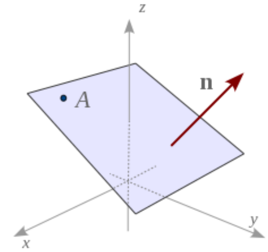
Worksheet 2

Topics: equation of a plane, distance formula, matrix multiplication.

An equation of the plane α passing through the point $A(x_0, y_0, z_0)$ perpendicular to the vector $\mathbf{n} = (A, B, C)$ is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Vector $\mathbf{n}$ is called a *normal* vector of the plane α.



1. Write an equation of the plane α.

a) The plane α passes through the point $A(1, 0, -2)$ perpendicular to the line $\mathbf{l}(t) = (2t, 1 - t, 3t)$.

Solution. In order to find an equation of a plane, we need to know its normal vector and a point on this plane. In our case, since the given line is perpendicular to the plane α, then a direction vector of this line can be taken as a normal vector of the plane. The vector $\mathbf{v} = (2, -1, 3)$ is a direction vector of the line. Hence, an equation of the plane is

$$2(x - 1) - 1(y - 0) + 3(z + 2) = 0.$$

After simplification, it becomes

$$2x - y + 3z + 4 = 0.$$

b) The plane α passes through the point $A(0, 4, 1)$ parallel to the plane $x - y + z = 100$.

Solution. If two planes α and β are parallel, then their normal vectors are parallel. In particular, if $\mathbf{n}$ is a normal vector of β, then $\mathbf{n}$ can serve as a normal vector of α. In our case, a normal vector of the plane $x - y + z = 100$ is $\mathbf{n} = (1, -1, 1)$ (take the coefficients standing by the variables x, y and z). Hence, the plane α has an equation

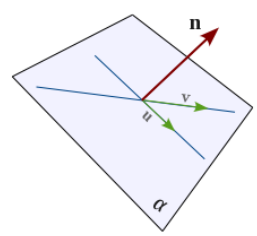
$$1 \cdot (x - 0) - 1 \cdot (y - 4) + 1 \cdot (z - 1) = 0.$$

After simplification,

$$x - y + z + 3 = 0.$$

c) The plane α contains the lines $\mathbf{c}_1(t) = (1 + 2t, 2 - 3t, 3)$ and $\mathbf{c}_2(t) = (-t, t + 3, 1 - 2t)$.

Solution. Since lines $\mathbf{c}_1$ and $\mathbf{c}_2$ lie in the plane α, their direction vectors $\mathbf{u} = (2, -3, 0)$ and $\mathbf{v} = (-1, 1, -2)$ are also contained in α. Hence, the cross product $\mathbf{u} \times \mathbf{v}$ is a vector perpendicular to α. We compute



$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ -1 & 1 & -2 \end{vmatrix} = (6-0)\mathbf{i} - (-4-0)\mathbf{j} + (2-3)\mathbf{k} = (6, 4, -1).$$

The point $A = \mathbf{c}_1(0) = (1,2,3)$ certainly lies in the plane. Hence, the plane α has an equation

$$6 \cdot (x-1) + 4 \cdot (y-2) - 1 \cdot (z-3) = 0,$$

which is

$$6x + 4y - z - 11 = 0.$$

Question. Why this solution would not work if the given two lines were parallel? How would you solve the problem in such case?

2. Determine if points $A(2,-1,-2)$, $B(1,2,1)$, $C(2,3,0)$, $D(5,0,-6)$ lie in the same plane.

Solution 1. *Idea*: we will find an equation of the plane α passing through the points A, B and C (see example 1.3.11, p.51 of the textbook) and check if point D satisfies this equation as well.

The vector $\overline{AB} \times \overline{AC}$ is normal to the plane α. We find

$$\begin{aligned} \overline{AB} &= (-1,3,3) \\ \overline{AC} &= (0,4,2) \\ \overline{AB} \times \overline{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 3 \\ 0 & 4 & 2 \end{vmatrix} = (6-12)\mathbf{i} - (-2-0)\mathbf{j} + (-4-0)\mathbf{k} = (-6,2,-4). \end{aligned}$$

Thus the plane α has an equation

$$-6 \cdot (x-2) + 2 \cdot (y+1) - 4 \cdot (z+2) = 0.$$

Now we check whether the coordinates of the point D satisfy this equation:

$$-6 \cdot (5-2) + 2 \cdot (0+1) - 4 \cdot (-6+2) = -18 + 2 + 16 = 0.$$

They do. Hence, D is contained in the same plane as A, B and C.

Solution 2 (for those who are familiar with linear algebra).

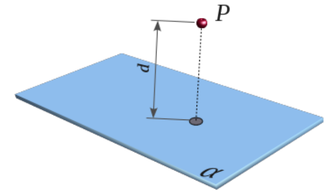
The points A, B, C and D lie in the same plane, if and only if the vectors $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ lie in the same plane. Three vectors lie in the same plane if and only if they are linearly dependent, and three vectors in a three-dimensional space are linearly dependent when the 3-by-3 determinant made up of the coordinates of these vectors is zero. We compute

$$\overline{AB} = (-1,3,3), \quad \overline{AC} = (0,4,2), \quad \overline{AD} = (3,1,-4)$$

$$\begin{vmatrix} -1 & 3 & 3 \\ 0 & 4 & 2 \\ 3 & 1 & -4 \end{vmatrix} = (-1) \cdot (-16-2) - 3 \cdot (0-6) + 3 \cdot (0-12) = 0.$$

So all four points are contained in the same plane.

The distance from a point $P(x_0, y_0, z_0)$ to a plane α defined by the equation $Ax + By + Cz + D = 0$ is equal to $d = \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$.



3. The plane α passes through the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. How far is it from the origin?

   Solution. Denote the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ by P, Q and R respectively. To find an equation of the plane α passing through these points, notice that the vector $\overline{PQ} \times \overline{PR}$ is normal to this plane. We compute

$$\overline{PQ} = (-1, 1, 0)$$
$$\overline{PR} = (-1, 0, 1)$$
$$\overline{PQ} \times \overline{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = (1-0)\mathbf{i} - (-1-0)\mathbf{j} + (0-(-1))\mathbf{k} = (1, 1, 1).$$

   Hence, the plane α has an equation

$$1 \cdot (x-1) + 1 \cdot (y-0) + 1 \cdot (z-0) = 0$$
$$x + y + z - 1 = 0$$

   and the distance from the origin point $O(0, 0, 0)$ to this plane is equal to

$$d = \frac{|1 \cdot 0 + 1 \cdot 0 + 1 \cdot 0 - 1|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}.$$

4. Find the distance between the planes $x - y + z = 2011$ and $x - y + z = 2012$.

   Solution. To find the distance between two given planes, we will pick a point P on the first plane and measure the distace from this point to the second plane.

   We plug $x = 0$, $y = 0$ into the first equation and obtain $z = 2011$. So the point $(0, 0, 2011)$ lies in the first plane. The distance from this point to the second plane is equal to

$$d = \frac{|1 \cdot 0 - 1 \cdot 0 + 1 \cdot 2011 - 2012|}{\sqrt{1^2 + (-1)^2 + 1^2}} = \frac{1}{\sqrt{3}}.$$

5. For a square matrix M, the matrix product $\underbrace{M \cdot M \dots M}_{n \text{ factors}}$ is denoted by M^n.

Compute $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}^{100}$.

Solution. We calculate first few powers of the given matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}^{2} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}^{3} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -8 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}^{4} = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{pmatrix};$$

and see the pattern . So we conclude[1] that

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}^{100} = \begin{pmatrix} 1 & 100 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-2)^{100} \end{pmatrix}.$$

[1]A more rigorous explanation would require using the method of mathematical induction.