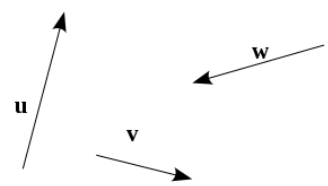
'Where shall I begin, please your Majesty?' he asked.
'Begin at the beginning,' the King said gravely, 'and go on till you come to the end: then stop.'

Lewis Carroll
Alice's Adventures in Wonderland

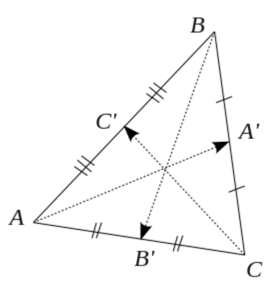
Worksheet 1

1. Given vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, draw the vectors

a) $\mathbf{u}+\mathbf{w}$; b) $\mathbf{u}+\mathbf{v}+\mathbf{w}$; c) $\mathbf{v}-\mathbf{w}$; d) $-\frac{1}{2}\mathbf{u}+2\mathbf{w}$; e) $\underbrace{\mathbf{u}-\mathbf{v}+\mathbf{u}-\mathbf{v}+\cdots+\mathbf{u}-\mathbf{v}}_{100}$



2. Let ABC be a triangle with the medians AA', BB' and CC'.
Prove that a) $\overline{AA'} = \frac{1}{2}(\overline{AB} + \overline{AC})$; b) $\overline{AA'} + \overline{BB'} + \overline{CC'} = \mathbf{0}$.



Solution.

a) We have $\overline{AA'} = \overline{AB} + \overline{BA'}$ and $\overline{AA'} = \overline{AC} + \overline{CA'}$. Adding these identities gives

$$2\overline{AA'} = \overline{AB} + \overline{AC} + \overline{BA'} + \overline{CA'}. \tag{1}$$

Since AA' is the median, $|BA'| = |CA'|$. Notice also that the vectors $\overline{BA'}$ and $\overline{CA'}$ are pointing in the opposite directions. Hence, the sum of $\overline{BA'}$ and $\overline{CA'}$ is zero. Then we can rewrite (1) as $2\overline{AA'} = \overline{AB} + \overline{AC}$. Dividing both sides by 2, we obtain the desired identity.

b) In part a) we established the identity

$$\overline{AA'} = \frac{1}{2}(\overline{AB} + \overline{AC}). \tag{2}$$

In the same way, we can prove that

$$\begin{aligned} \overline{BB'} &= \frac{1}{2}(\overline{BA} + \overline{BC}) \\ \overline{CC'} &= \frac{1}{2}(\overline{CA} + \overline{CB}). \end{aligned} \tag{3}$$

Summing (2) and (3), we get

$$\overline{AA'} + \overline{BB'} + \overline{CC'} = \frac{1}{2}(\overline{AB} + \overline{AC}) + \frac{1}{2}(\overline{BA} + \overline{BC}) + \frac{1}{2}(\overline{CA} + \overline{CB}).$$

We rearrange the terms in this identity:

$$\overline{AA'} + \overline{BB'} + \overline{CC'} = \frac{1}{2}[(\overline{AB} + \overline{BA}) + (\overline{AC} + \overline{CA}) + (\overline{BC} + \overline{CB})].$$

Each of the sums standing in parentheses is zero. Hence $\overline{AA'} + \overline{BB'} + \overline{CC'} = \mathbf{0}$.

3. $ABCD$ is a parallelogram with the vertices $A(-6,-1)$, $B(1,2)$, $D(-3,-2)$. Find the coordinates of the point C.

Solution. Since $ABCD$ is a parallelogram, then $|AD| = |BC|$ and the lines AD and BC are parallel. It implies that the vectors $\overline{AD}$ and $\overline{BC}$ point in the same direction and have equal magnitudes (make a sketch). Therefore, they are equal. We have

$$\overline{AD} = \underbrace{(-3,-2)}_{D} - \underbrace{(-6,-1)}_{A} = (3,-1).$$

Since

$$\overline{AC} = \overline{AB} + \overline{BC}$$

and $\overline{AB} = \underbrace{(1,2)}_{B} - \underbrace{(-6,-1)}_{A} = (7,3)$, then

$$\overline{AC} = (7,3) + (3,-1) = (10,2).$$

So we obtain

$$C = A + \overline{AC} = (-6,-1) + (10,2) = (4,1).$$

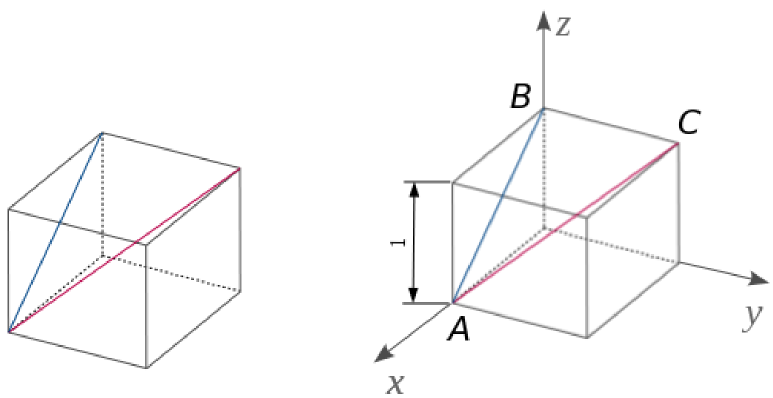
4. Determine if the triangle ABC with $A = (1, 2, 3)$, $B = (2, 1, 3)$, $C = (3, 1, 2)$, is obtuse-angled.

Solution.

$$\cos\angle CAB = \frac{\overline{AB}\cdot\overline{AC}}{\|\overline{AB}\|\cdot\|\overline{AC}\|} = \frac{(1,-1,0)\cdot(2,-1,-1)}{\sqrt{2}\cdot\sqrt{6}} = \frac{3}{\sqrt{12}}$$
$$\cos\angle ABC = \frac{\overline{BA}\cdot\overline{BC}}{\|\overline{BA}\|\cdot\|\overline{BC}\|} = \frac{(-1,1,0)\cdot(1,0,-1)}{\sqrt{2}\cdot\sqrt{2}} = -\frac{1}{2}$$
$$\cos\angle BCA = \frac{\overline{CB}\cdot\overline{CA}}{\|\overline{CB}\|\cdot\|\overline{CA}\|} = \frac{(-1,0,1)\cdot(-2,1,1)}{\sqrt{2}\cdot\sqrt{6}} = \frac{3}{\sqrt{12}}$$

Since $\cos\angle ABC < 0$, then $\angle ABC > \frac{\pi}{2}$. So the triangle ABC is obtuse-angled.

5. Find the angle between the main diagonal of a cube and the diagonal of one of its faces.



Solution. Let a be the edge length of the cube. Set up a coordinate system as shown on the picture. In this coordinate system, points A, B, C have coordinates $(a, 0, 0)$, $(0, 0, a)$ and $(0, a, a)$ respectively. Then $\overline{AB} = (-a, 0, a)$, $\overline{AC} = (-a, a, a)$ and we find

$$\cos\angle BAC = \frac{\overline{AB}\cdot\overline{AC}}{\|\overline{AB}\|\cdot\|\overline{AC}\|} = \frac{2a^2}{\sqrt{2a^2}\cdot\sqrt{3a^2}} = \frac{\sqrt{6}}{3}$$
$$\angle BAC = \arccos\frac{\sqrt{6}}{3}$$

6. (The Converse Pythagorean Theorem)

Let ABC be a triangle such that $|AB|^2 + |BC|^2 = |AC|^2$. Show that ABC must be a right triangle.

Solution. We are going to show that the angle $\angle ABC$ is right.

Since $\overline{AC} = \overline{AB} + \overline{BC}$, then

$$\begin{aligned} \|\overline{AC}\|^2 = \overline{AC} \cdot \overline{AC} = (\overline{AB} + \overline{BC}) \cdot (\overline{AB} + \overline{BC}) &= \overline{AB} \cdot \overline{AB} + \overline{BC} \cdot \overline{AB} + \overline{AB} \cdot \overline{BC} + \overline{BC} \cdot \overline{BC} \\ &= \|\overline{AB}\|^2 + \|\overline{BC}\|^2 + 2\overline{BC} \cdot \overline{AB}. \end{aligned} \tag{4}$$

Since it is given that $|AC|^2 = |AB|^2 + |BC|^2$, then we can rewrite (4) as

$$\|\overline{AC}\|^2 = \|AC\|^2 + 2\overline{BC} \cdot \overline{AB}.$$

Therefore, $2\overline{BC} \cdot \overline{AB} = 0$.

$$\begin{aligned} \cos \angle ABC &= \frac{\overline{BC} \cdot \overline{BA}}{\|\overline{BC}\| \cdot \|\overline{BA}\|} = \frac{\overline{BC} \cdot (-\overline{AB})}{\|\overline{BC}\| \cdot \| - \overline{AB}\|} = \frac{0}{\|\overline{BC}\| \cdot \| - \overline{AB}\|} = 0 \\ \angle ABC &= \arccos 0 = \frac{\pi}{2}. \end{aligned}$$