0. Introduction

An operad is an algebraic gadget allowing one to keep track on collections of composable $n$-ary operations for varying $n$. In a bit more details, the data determining an operad consists of a sequence of objects $\{O(n)\}_{n \geq 1}$ of a symmetric monoidal category and the morphisms

$$O(n) \times O(k_1) \times \cdots \times O(k_n) \to O(k_1 + \cdots + k_n)$$

defined for all integer $n \geq 1$ and $k_i \geq 1$ subject to some natural equivariance and associativity conditions. Here, elements of $O(n)$ can be thought of as $n$-ary operations, and the given mappings are supposed to represent superposition rules for them. As it turns out, such a gadget becomes a useful tool in handling different kinds of algebraic structures arising, in particular, in topology and geometry. On the purely algebraic side, we will see that it will be quite a typical situation, when a single (and in reasonable cases, rather tangible) object - an operad - is put in charge of controlling the whole class of algebras (say, associative, commutative, Lie or Gerstenhaber). Then transforming the original operad (perturbing, dualizing or making a cofibrant replacement - whatever it would mean in a specific case) will reflect in a coherent change of the original algebraic structure. For instance, this is how one passes from the classical algebras to their homotopy counterparts.

From another point of view, operads generalize the notion of an associative algebra. The latter arises naturally as one studies the structure of endomorphisms of a vector space. Given an associative algebra $A$ (a group algebra, for instance), it is useful to study maps from $A$ to the endomorphism algebra $\text{End}(V)$ of some vector space $V$ - this constitutes the rich subject of the representation theory. Similarly, given an operad $A$ defined in a symmetric monoidal category $C$ and an object $V \in C$, one can benefit from investigating "representations" $A \to \text{End}_V$, where $\text{End}_V$ is the so-called endomorphism operad naturally associated with the object $V$. Constructing such a map can also be viewed as endowing $V$ with some extra structure encoded by the operad in the same way as giving a morphism $A \to \text{End}(V)$ equips a vector space $V$ with an $A$-module structure.
Operads in a fixed symmetric monoidal category $C$ themselves form a category with a rather rich structure. In particular, under relatively mild condition there is a natural transfer of a model structure on $C$ to the one on the category of operads in $C$. This provides a way to study operads and algebras over operads by the means of abstract homotopy theory [BM03]. For example, there is a way to define homology functors for algebras over operads as certain derived functors in the sense of Quillen [GJ94].

One of the central objects of interest in the present paper are algebras over modular operads. Those are operads with some extra structure which, in view of the heuristic image suggested above, allows plugging the output cord of an operation into an input slot of the same operation, thus closing some sort of a circuit. The prototypical example of such an operad arises in a geometric context as the operad of moduli spaces of punctured Riemann surfaces (thus the name), where modular operads allow one to keep track on combinatorial data associated with the moduli spaces. To a large extent, it is due to this and similar examples that operads proved to be useful in physics, in conformal and string field theories, in particular, where the moduli spaces of Riemann surfaces with punctures or marks play an important role.

As a short historical remark, we note that the concept of an operad in its modern form emerged in topology in the 1960’s, and its origins can be traced back to the works of S. MacLane [ML63], J. Stasheff [Sta63a][Sta63b], M. Boardman and R. Vogt [BV73], though the term *operad* \(^1\) and its formal definition were first introduced by J. P. May in his work on iterated loop spaces [May72]. Originally developed for the purposes of topology (the theory iterated loops spaces, in particular), later in 1990’s operads found fruitful applications in deformation theory, mathematical physics, moduli theory, homological algebra. Here, the reader can be referred, for example, to works of E. Getzler, J. Jones [GJ94], V. Ginzburg, M. Kapranov [GK94], E. Getzler, M. Kapranov [GK95], M. Kontsevich [Kon99], [KS00], R. Kauffman [Kau04], M. Gerstenhaber, A. Voronov [GV94], T. Kimura, J. Stasheff, A. Voronov [KSV95].

The purpose of this paper is twofold. On the one hand, we would like to give an exposition of some basic constructions and methods of the operad theory, and on the other hand, as an application, we would like to give an account of the following result of S. Barannikov [Bar07] that establishes a connection between certain algebraic structures associated with modular operads and Batalin-Vilkovisky geometry:

**Theorem 0.1.** For a differential graded modular operad $\mathcal{P}$, there is a bijection between the solutions of the quantum master equation of Batalin-Vilkovisky geometry on the affine $\mathcal{P}$-manifold and structures of algebras over the Feynman transform of $\mathcal{P}$ on a differential graded vector space with symmetric inner product.

This result is analogous to the bijections well-known from the classical deformation theory:

$$\left\{ \begin{array}{c}
A_\infty\text{-structures} \\
on \text{a graded vector space } V
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{c}
differentials \text{ on the} \\
tensor \text{ coalgebra } T(V[-1])
\end{array} \right\}$$

$$\left\{ \begin{array}{c}
L_\infty\text{-structures} \\
on \text{a graded vector space } V
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{c}
differentials \text{ on the} \\
symmetric \text{ coalgebra } S(V[-1])
\end{array} \right\}$$

where the structures on the right hand side are obtained as solutions of the classical (as opposed to quantum) master equation, also known as the Maurer-Cartan equation.

The paper is organized as follows. In the first section we fix notation and give some basic definitions, facts and examples regarding classical operads. Cyclic and modular operads are discussed in section 2. In section 3 we describe some standard ways of constructing

\(^1\) Which itself is a portmanteau word coming from “operations” and “monad” (cf. remark 1.6)
cofibrant resolutions of operads in the topological (Boardman-Vogt resolution) and differential graded (cobar resolution) settings, and discuss the structure of algebras over the cobar construction. In section 4 we discuss an analogue of the cobar construction defined for modulars operad (Feynman transform). Section 5 is devoted for explaining the above result of S.Barannikov. We contemplate some directions for further research in section 6.

1. OPERADS: DEFINITIONS AND EXAMPLES

The following fundamental example precedes the general definition.

Example 1.1. Let $X$ be a topological space. Consider the spaces of continuous functions $\mathcal{O}(n) = Map(X^n, X)$ for $n \geq 1$ taken together with the mappings

\begin{align*}
\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) &\rightarrow \mathcal{O}(n + m - 1) \\
(f,g) &\mapsto f \circ_i g
\end{align*}

defined for each $1 \leq i \leq n$ by

\[(f \circ_i g)(x_1, x_2, \ldots, x_{n+m-1}) = f(x_1, \ldots, x_{i-1}, g(x_i, \ldots, x_{i+m-1}), x_{i+1}, \ldots, x_{n+m-1})\]

and the action of the symmetric group $\Sigma_n$ on $\mathcal{O}(n)$ by permutations of the inputs.

Alternatively, instead of working with mappings (1), we may invoke composition laws

\[(2) \quad \gamma_{k;n_1,\ldots,n_k} : \mathcal{O}(k) \times \mathcal{O}(n_1) \times \cdots \times \mathcal{O}(n_k) \rightarrow \mathcal{O}(n_1 + \cdots + n_k),\]

where for positive integer $k$ and $n_i$’s, the map $\gamma_{k;n_1,\ldots,n_k}$ sends a tuple $(f, g_1, \ldots, g_n)$ to the composition $f(g_1(-), \ldots, g_k(-))$. Further, we will omit subscripts for $\gamma$, whenever there is no risk of confusion.

One observes then that for $\gamma$’s the following properties hold.

(a) Associativity: the diagram

\[
\begin{array}{ccc}
\mathcal{O}(k) \times \mathcal{O}(n_1) \times \cdots \times \mathcal{O}(n_k) & \xrightarrow{id \times \gamma \times \cdots \times \gamma} & \mathcal{O}(k) \times \mathcal{O}(m_1) \cdots \times \mathcal{O}(m_k) \\
\mathcal{O}(m_1,1) \times \cdots \times \mathcal{O}(m_{k,n_k}) & \xrightarrow{\gamma \times id} & \mathcal{O}(m)
\end{array}
\]

commutes. Here, $m_i = \sum_j m_{i,j}$, $n = \sum_j n_i$ and $m = \sum_j m_i$.

(b) Equivariance: the morphism $\gamma$ is equivariant with respect to the symmetric group action. That means, the group $\Sigma_k \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$ acts on the left-hand side of (2) and maps naturally to $\Sigma_{n_1+\cdots+n_k}$ acting on the right-hand side;

(c) Existence of unit: there is an element $e \in \mathcal{O}(1)$ called the unit such that $\gamma(e, f) = f$ and $\gamma(f, e, \ldots, e) = f$ for any $f \in \mathcal{O}(k)$. Specifically in our case, $e = id_X$.

Given the above data, we say that a topological endomorphism operad is defined.

This example serves as a prototype for a more general construction:

Definition 1.2. Let $(C, \times, I)$ be a symmetric monoidal category with all colimits. An operad $\mathcal{O}$ in the category $C$ consists of

1. a collection of objects $\{\mathcal{O}(n)\}_{n \geq 1}$, where each $\mathcal{O}(n)$ is endowed with an action the symmetric group $\Sigma_n$, and

2. morphisms $\gamma_{k;n_1,\ldots,n_k}$ defined as in (2) for all positive integers $k$ and $n_1, \ldots, n_k$ subject to the conditions (a)-(c) listed above.
**Definition 1.3.** Let $\mathcal{P}$ and $\mathcal{Q}$ be operads in a category $\mathbf{C}$. A morphism $\mathcal{P} \xrightarrow{f} \mathcal{Q}$ of operads is a sequence of equivariant morphisms $\{\mathcal{P}(n) \xrightarrow{f_n} \mathcal{Q}(n)\}_{n \geq 1}$ such that $f_1$ maps the unit element of $\mathcal{P}$ to the unit element of $\mathcal{Q}$ and

$$f_{n_1+\ldots+n_k}(\gamma(g,h_1,\ldots,h_k)) = \gamma(f_k(g),f_{n_1}(h_1),\ldots,f_{n_k}(h_k))$$

for all $h_i$'s in $\mathcal{O}(n_i)$ and $g \in \mathcal{O}(k)$. In other words, each $f_n$ is required to commute with all suitable structure morphisms $\gamma$.

**Remark 1.4.** Omitting any reference to the symmetric group action in definition 2.1 gives rise to a more general notion - a non-$\Sigma$ operad and neglecting the existence of unit axiom, we obtain a non-unital operad. For any non-$\Sigma$ operad $\mathcal{P}$, there is associated symmetric operad $\mathcal{P}$ with $\mathcal{P}(n) := \mathcal{P}(n) \times \Sigma_n$, and any non-unital operad $\mathcal{Q}$ can be given a unit in the canonical way.

**1.5. Operads and $\Sigma$-modules.** Alternatively, an operad in a symmetric monoidal category $(\mathbf{C}, \otimes, I)$ can be defined as a monoid in the category of $\Sigma$-modules in $\mathbf{C}$. The category of (right) $\Sigma$-modules $\text{Mod}C\Sigma$ is defined as the functor category $\mathbf{C}^{\Sigma^{op}}$, where $\Sigma$ is the groupoid of finite sets $[n] = \{1, \ldots, n\}$ and all bijections. In other words, a $\Sigma$-module $\mathcal{A}$ can be thought of as a collection $\{\mathcal{A}(n)\}_{n \geq 1}$ of objects in $\mathbf{C}$, where each $\mathcal{A}(n)$ is equipped with a right action of $\Sigma_n$. The monoidal product in $\text{Mod}C\Sigma$ is defined as follows. Any $\mathcal{A} \in \text{Mod}C\Sigma$ induces an endofunctor $F_\mathcal{A} : \mathbf{C} \to \mathbf{C}$ (the Schur functor) via

$$F_\mathcal{A}(X) = \coprod_{n \geq 1} \mathcal{A}(n) \otimes_{\Sigma_n} X^\otimes n.$$  

As it turns out [Mar08], for $\mathcal{A}, \mathcal{B} \in \text{Mod}C\Sigma$, the composition of Schur functors $F_\mathcal{B} \circ F_\mathcal{A}$ is itself a Schur functor induced by some $\Sigma$-module, which we take to be the product of $\mathcal{A}$ and $\mathcal{B}$ in $\text{Mod}C\Sigma$. Explicit formulas for the product and the verification of associativity can be found in [GJ94].

The unit element for this monoidal structure is the $\Sigma$-module $I$ defined by letting

$$I[n] = \begin{cases} I, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

where $0$ is the initial object in $\mathbf{C}$, whose existence is now assumed. Knowing explicit formulas for the product, one can show now that the axioms for a monoid in $\text{Mod}C\Sigma$ can be translated into the definition of an operad given above.

**Remark 1.6.** If $\mathcal{A}$ is an operad in $\Sigma \mathbf{C}$, the corresponding Schur functor $F_\mathcal{A}$ is going to be a monoidal object in the category of endofunctors on $\mathbf{C}$, that is, a monad in $\mathbf{C}$. The category of algebras over the monad $F_\mathcal{A}$ in is naturally isomorphic to the category of algebras over the operad $\mathcal{A}$ (cf. definition 1.19).

**Example 1.7.** Let $k$ be a field and $V$ be a (unital) $k$-algebra. The collection $\{\mathcal{P}(n)\}_{n \geq 1}$ of $k$-vector spaces, where $\mathcal{P}(1) = V$ and $\mathcal{P}(n) = 0$ for $n > 1$, with the composition law

$$\gamma_{1,1} : \mathcal{P}(1) \otimes \mathcal{P}(1) \to \mathcal{P}(1)$$

$$v \otimes w \mapsto v \cdot w$$

and the trivial symmetric group action forms an operad in the category $\text{Vect}_k$ of $k$-vector spaces.

Conversely, given any operad $\mathcal{Q}$ in $\text{Vect}_k$, $\mathcal{Q}(1)$ has a natural structure of a $k$-algebra. Thus operads in the category of vector spaces can be thought of as a generalization of associative algebras.
**Example 1.8.** Let \((\mathbf{C}, \otimes, I)\) be a closed symmetric monoidal category. The endomorphism operad \(\text{End}_{\mathbf{C}}(X)\) for an object \(X \in \mathbf{C}\) is the operad in \(\mathbf{C}\) defined by taking

\[
\text{End}_{\mathbf{C}}(X) := \hom_{\mathbf{C}}(X^{\otimes n}, X)
\]

with the structure morphisms

\[
\gamma : \text{End}_{\mathbf{C}}(k) \otimes \text{End}_{\mathbf{C}}(n_1) \otimes \cdots \otimes \text{End}_{\mathbf{C}}(n_k) \to \text{End}_{\mathbf{C}}(n_1 + \cdots + n_k)
\]

\[
f \otimes g_1 \otimes \cdots \otimes g_k \mapsto f(g_1(-) \otimes \cdots \otimes g_k(-)).
\]

The action of \(\sigma \in \Sigma_n\) on \(\text{End}_{\mathbf{C}}(n)\) is defined via

\[
(f \cdot \sigma)(x_1 \otimes \cdots \otimes x_n) = f(x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)})
\]

and the unit element is \(\text{id}_X \in \hom_{\mathbf{C}}(X, X)\).

**1.9. Operads and trees.** Many operadic constructions are closely connected to combinatorics of graphs and trees, in particular. In a sense, this is due to the fact that an oriented finite tree conveniently represents a way of composing algebraic operations. In the current and the next two following subsections, we will make these ideas more precise. First, as an example, we will describe the tree operad defined in the category of sets. Then we will show how to construct a free operad from the data of a \(\Sigma\)-module by taking certain coproducts over the groupoid of trees. This will be the first, but not the last, construction of this type that we will encounter. We will conclude by giving a characterization of operads as algebras over the monad of rooted trees.

Before we discuss all that, we need to introduce (or rather fix notation and terminology) a few notions.

**Definition 1.10.** A graph \(\Gamma\) is a triple \((\text{Flag}(\Gamma), \lambda, \sigma)\), where \(\text{Flag}(\Gamma)\) is a finite set, whose elements are called flags or half-edges, \(\lambda\) is a partition of \(\text{Flag}(\Gamma)\), and \(\sigma\) is an involution acting on \(\text{Flag}(\Gamma)\).

By vertices of the graph we mean the blocks of the partition \(\lambda\). The set of these blocks will be denoted by \(\text{Vert}(\Gamma)\). We say that flags \(x\) and \(y\) meet is they belong to the same block of \(\lambda\). For \(v \in \text{Flag}(\Gamma)\), we let \(\text{Leg}(v)\) to be the block of \(\lambda\) containing \(v\), and we set \(n(v) := |\text{Leg}(v)|\).

The pairs of flags that form a 2-cycle with respect to \(\sigma\) will be referred to as edges and the fixed points of \(\sigma\) will be called legs of the graph. We denote those subsets of \(\text{Flag}(\Gamma)\) by \(\text{Edge}(\Gamma)\) and \(\text{Leg}(\Gamma)\) respectively. Our definition of a graph is slightly different from the most common one, exactly because for our purposes graphs should have not just edges and vertices, but legs as well.

In a pretty evident way one introduces the usual notions of finiteness, cyclicity, connectedness and orientation for graphs and constructs a geometric realization of a graph.

**Definition 1.11.** A morphism \(f : \Gamma_0 \to \Gamma_1\) of graphs \(\Gamma_0 = (\text{Flag}(\Gamma_0), \lambda_0, \sigma_0), \Gamma_1 = (\text{Flag}(\Gamma_1), \lambda_1, \sigma_1)\) is an injection \(f^* : \text{Flag}(\Gamma_1) \to \text{Flag}(\Gamma_0)\) subject to conditions

1. \(\sigma_0 \otimes f^* = f^* \otimes \sigma_1\);
2. \(\sigma_0\) acts freely on \(\text{Flag}(\Gamma_0) \setminus \text{im}(f^*)\). Geometrically, it means that \(\Gamma_1\) is obtained from \(\Gamma_0\) by contracting a subset \(J\) (possibly empty) of its edges. In such a case we will write \(\Gamma_1 = \Gamma_0/J\).
3. Two flags \(a, b\) in \(\Gamma_1\) meet if and only if there is a sequence \((x_0, \ldots, x_k)\) of flags in \(\Gamma_0\) such that \(f^*(a) = x_0, f^*(b) = x_k\) and \(\sigma_0(x_{i-1})\) meets \(x_i\) for all \(1 \leq i \leq k\).
Definition 1.12. A \textit{rooted non-planar tree} is a finite, non-empty, connected, oriented acyclic graph with the property that for each vertex \( v \in \text{Vert}(\Gamma) \), there is at least one incoming and exactly one outgoing flag.

We denote by \( \text{In}(v) \) the set of flags meeting at a vertex \( v \) and oriented towards \( v \). Any tree has a unique outgoing leg, which we will call the root of the tree, and several incoming legs called the leaves.

Example 1.13. Let \( \text{Tree}(n) \) be the set of rooted oriented trees with \( n \) leaves labeled by numbers from 1 to \( n \). Labeling allows us to equip \( \text{Tree}(n) \) with an action of a symmetric group \( \Sigma_n \), which permutes the labels. To give \( \{\text{Tree}(n)\}_{n \geq 1} \) the structure of a set-valued operad, we define the composition law by letting \( \gamma(T, T_1, \ldots, T_k) \) be the tree obtained by grafting the root of \( T_i \) to the \( i \)-th leaf of \( T \) for \( 1 \leq i \leq l \) with the naturally induced labeling and orientation.

For instance,

\[
\begin{pmatrix}
1 & 2 & 1 & 2 \\
, & & & \\
1 & 2 & 3 & 1 & 2 & 3
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}
\]

The unit of the tree operad is the degenerate tree having only one edge and no vertices.

1.14. \textit{Free operads}. We describe now the free operad functor \( L \), which is left adjoint to the forgetful functor

\[
U : \{\text{Operads in C}\} \to \text{Mod}_C-\Sigma.
\]

We start by extending a \( \Sigma \)-module \( E : \Sigma^{op} \to C \) to the contravariant functor on a category \( \text{fSets} \) of all finite non-empty sets. Namely, for a set \( S \) of cardinality \( n \), we take

\[
E(S) := E(n) \otimes_{\Sigma_n} \text{Iso}([n], S),
\]

where \( \text{Iso}([n], S) \) is the set of all bijections \( [n] \to S \), on which \( \Sigma_n \) acts naturally from the left. Now, let \( \text{Tree}_n \) be the category, whose objects are pairs \( (T, l) \), where \( T \) is a rooted non-planar tree and \( l \) is a bijection

\[
l : \{\text{leaves of } T\} \to [n].
\]

In this category morphisms from \( (T, l) \) to \( (T', l') \) are graph isomorphisms \( T \to T' \) preserving orientations of flags. For \( T \in \text{Tree}_n \), define

\[
E(T) := \bigotimes_{v \in \text{Vert}(T)} E(\text{In}(v)).
\]

and for \( n \geq 1 \),

\[
\Psi(E)(n) := \text{colim}_{T \in \text{Tree}_n} E(T).
\]

Since \( \text{Tree}_n \) is a groupoid, \( \Psi(E)(n) \) can be (non-canonically) identified with the coproduct of \( E(T) \)'s for \( T \) running over isomorphism classes of trees with \( n \) leaves.

Theorem 1.15. ([Mar08], Theorem 32) \textit{There exists a non-unital operad structure on the \( \Sigma \)-module} \( \Psi(E) = \{\Psi(E)(n)\}_{n \geq 1} \), \textit{which turns it into a free non-unital operad generated by} \( E \).
Having defined a free non-unital operad $\Psi(E)$, we obtain a free operad $L(E)$ by formally adjoining the unit element to $\Psi(E)(1)$.

1.16. **The monad of rooted trees.** By a standard categorical argument [ML98], the composition of adjoint functors

$$T : \text{Mod}_{C \cdot \Sigma} \xrightarrow{L} \{\text{Operads in } C\} \xrightarrow{U} \text{Mod}_{C \cdot \Sigma}$$

is a monad in the category of $\Sigma$-modules in $C$, which we call the monad of rooted trees. The following statement holds.

**Theorem 1.17.** ([GJ94] Proposition 1.12) **Operads in a symmetric monoidal category $C$ are precisely algebras over the monad $T$.**

To a large extent, the proof amounts to showing that for an operad $P$ and a rooted $n$-tree $T$, there is a map $\gamma_T : P(T) \to P(n)$ compatible with the composition laws on $P$ and grafting operation on trees.

1.18. **Algebras over operads.** In most cases, the point of main interest is not an operad itself, but rather an object equipped with an action of the operad. In this subsection we define what does it mean for an object of a symmetric monoidal category to have an action of an operad and show what kind of operads govern some classical algebras.

Giving a $k$-vector space $V$ a structure of a module over a $k$-algebra $A$ amounts to constructing a $k$-algebra homomorphism $A \to \text{End}_k(V)$. By analogy, one introduces the following definition.

**Definition 1.19.** Let $O$ be an operad in a closed symmetric monoidal category $C$ and $V$ be an object in $C$. We say that $V$ is an algebra over $O$ (or simply $O$-algebra) if there is a morphism $O \to \text{End}_V$.

**Remark 1.20.** By adjointness of the monoidal product with the internal hom-functor in $C$, this is the same as giving a sequence of morphisms $O(n) \otimes V^\otimes n \to V$ satisfying certain compatibility conditions.

**Remark 1.21.** Let $P$ be an operad in a closed symmetric monoidal category $C$. Then algebras over $P$ form a category with morphisms being $C$-morphisms compatible with the $P$-algebra structure. That means, $h : V \to W$ is a morphism in the category of $P$-algebras iff for any $n \geq 1$, the below diagram commutes in $C$:

$$
\begin{array}{c}
\P(n) \otimes V^\otimes n \xrightarrow{id \otimes h \cdots \otimes h} V \\
\P(n) \otimes W^\otimes n \xrightarrow{h} W
\end{array}
$$

Given a $\Omega$-algebra $X$ in $C$, a morphism of operads $f : \P \to \Omega$ induces a $\P$-algebra structure on $X$ via the composition $\P \to \Omega \to \text{End}_X$. Moreover, this correspondence is natural, that is, it extends to a functor $f^* : \Omega$-algebras $\to \P$-algebras.

**Example 1.22.** Any $X \in C$ is trivially an $\text{End}_X$-algebra, and for any operad $P$ in $C$, there is a free $P$-algebra generated by $X$. Namely, it is $F_P(X)$, where $F_P$ is the Schur functor (cf. subsection 1.5).

**Example 1.23.** For $n \geq 1$, take $\text{Ass}(n)$ to be the group algebra $k[\Sigma_n]$ considered as a $k$-vector space with an obvious $\Sigma_n$-action. We will turn the collection $A = \{\text{Ass}(n)\}_{n \geq 1}$ into...
an operad, called the \textit{associative operad}, by defining the composition maps \(\gamma\) as follows. Consider first the natural embedding

\[ \epsilon_{n_1, \ldots, n_k} : \Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \to \Sigma_{n_1 + \cdots + n_k} \]

and on generators \(\tau \in \Sigma_k, \sigma_i \in \Sigma_i (1 \leq i \leq k)\), take \(\gamma(\tau, \sigma_1, \ldots, \sigma_k)\) to be

\[ \epsilon_{n_1, \ldots, n_k}(\sigma_{\tau(1)}, \ldots, \sigma_{\tau(k)}). \]

Any algebra \(V\) over \(Ass\) has a structure of an associative \(k\)-algebra. Indeed, given the maps \(\theta_n : k[\Sigma_n] \to \text{Hom}(V^\otimes n, V)\), we can define a multiplication in \(V\) by letting the product of \(x_1, \ldots, x_n\) be \((\theta_n(e_n))(x_1 \otimes \cdots \otimes x_n)\), where \(e_n\) is the identity element in \(\Sigma_n\). Proving associativity of this product amounts to showing that for \(\mu = \theta_2(e_2) : V^\otimes 2 \to V\) the equation \(\mu(\mu \otimes \text{id}_V) = \mu(\text{id}_V \otimes \mu)\) holds. In our case,

\[ \mu(\mu \otimes \text{id}_V) = \theta_2(e_2)((\theta_2(e_2) \otimes \theta_1(e_1)) \]

\[ = \gamma(\theta_2(e_2), \theta_2(e_2), \theta_1(e_1)) \]

\[ = \theta_2(\gamma(e_2, e_2, e_1)) = \theta_2(\gamma(e_2, e_1, e_2)) \]

\[ = \gamma(\theta_2(e_2), \theta_1(e_1), \theta_2(e_2)) \]

\[ = \theta_2(e_2)(\theta_1(e_1) \otimes \theta_2(e_2)) = \mu(\text{id}_V \otimes \mu). \]

The converse is also true: any associative \(k\)-algebra \(V\) is an algebra over \(Ass\). Indeed, a structure morphism \(\theta : Ass \to \mathcal{E}nd_V\) for \(V\) can be defined by

\[ \theta_n : k[\Sigma_n] \to \text{Hom}(V^\otimes n, V) = \mathcal{E}nd_V(n) \]

\[ \sigma \mapsto (x_1 \otimes \cdots \otimes x_n \mapsto x_{\sigma(1)} \cdots x_{\sigma(n)}). \]

\textbf{Example 1.24.} The \textit{commutative operad} \(\Com\) is defined in \(\text{Vect}_k\) by taking \(\Com(n) = k\) with the trivial symmetric group action and the composition law \(\gamma\) is assumed to be nothing but the multiplication in \(k\). Algebras over \(\Com\) are precisely the associative commutative \(k\)-algebras. The product of elements \(x_1, \ldots, x_n\) in a \(\Com\)-algebra \(V\) with the structure morphism \(\theta : \Com \to \mathcal{E}nd_V\) is meant to be \(\theta_n(1)(x_1 \otimes \cdots x_n)\). Associativity of this product can be proved in essentially the same way as in the previous example and commutativity follows from the following computation:

\[ x_1x_2 = \theta_2(1)(x_1 \otimes x_2) = \theta_2(\tau \cdot 1)(x_1 \otimes x_2) = \tau \cdot \theta_2(1)(x_1 \otimes x_2) = \theta_2(1)(x_2 \otimes x_1) = x_2x_1, \]

where \(\tau = (12) \in \Sigma_2\).

\textbf{1.25. Operads defined by generators and relations.} We list a few definitions, which are obvious generalizations of the corresponding notions existing for associative algebras.

\textbf{Definition 1.26.} An operad \(\mathcal{P}\) in an abelian category \(\mathcal{C}\) is a called a \textit{suboperad} of the operad \(\mathcal{O}\) if there is a morphism \(\mathcal{P} \xrightarrow{f} \mathcal{O}\) such that for each \(n \geq 1, f_n : \mathcal{P}(n) \to \mathcal{O}(n)\) is an inclusion.

\textbf{Definition 1.27.} Let \(\mathcal{O}\) be an operad in an abelian category. A collection \(\mathcal{J} = \{\mathcal{J}(n)\}_{n \geq 1}\) of \(\Sigma_n\)-invariant subobjects \(\mathcal{J}(n) \subset \mathcal{O}(n)\) such that

\[ \gamma(f, g_1, \ldots, g_n) \in \mathcal{J}(k_1 + \cdots + k_n), \]

whenever \(f \in \mathcal{J}(n)\) or \(g_i \in \mathcal{J}(k_i)\) for some \(1 \leq i \leq n\) is a called an \textit{ideal} of \(\mathcal{O}\) and the collection \(\{\mathcal{O}(n)/\mathcal{J}(n)\}_{n \geq 1}\) with the composition laws induced from \(\mathcal{O}\) is called the \textit{quotient operad}.

Let \(E\) be \(\Sigma\)-module and \(\mathcal{R} \subset L(E)\) is an ideal of the free operad \(L(E)\). The quotient operad \(L(E)/\mathcal{R}\) is said to be generated by \(E\) with relations \(\mathcal{R}\).
Example 1.28. The operad $\mathcal{L}ie$ governing Lie algebras over a fixed field $k$ is generated by the $\Sigma$-module $E_{\mathcal{L}ie} = \{E_{\mathcal{L}ie}(n)\}_{n \geq 1}$ with
\[
E_{\mathcal{L}ie}(n) := \begin{cases} k \cdot \beta, & n = 2 \\ 0, & n \neq 2, \end{cases}
\]
where the right action of $\Sigma_n$ on the generator $\beta$ is
\[
\beta \cdot \sigma = \text{sgn}(\sigma) \cdot \beta.
\]
The ideal of relations $\mathcal{R}$ is generated by an element in $L(E_{\mathcal{L}ie})(3)$ representing the Jacobi identity:
\[
\beta(\beta \otimes \text{id}) + \beta(\beta \otimes \text{id})c + \beta(\beta \otimes \text{id})c^2 = 0.
\]
Here, $c \in \Sigma_3$ is a cyclic permutation $(1, 2, 3) \mapsto (2, 3, 1)$.

Remark 1.29. Similarly, one can find presentations for the associative and commutative operads with generators being $\Sigma$-modules $E = \{E(n)\}_{n \geq 1}$ non-trivial only for $n = 2$, and relations being ideals generated by elements in $E(3)$. Linear (that is, $\text{Vect}_k$-valued) operads with this property are called quadratic, based on an analogy with quadratic algebras [PP05].

Quadratic operads were studied by V. Ginzburg and M. Kapranov in the seminal work [GK94], where they introduced the notion of the quadratic dual and koszulness for operads. A quadratic dual for a quadratic operad $\mathcal{P}$ is the $k$-linear operad $\mathcal{P}^!$, whose generators are linear duals of generators of $\mathcal{P}$ and relations are annihilators of the relations of $\mathcal{P}$ in the dual space. For instance, it is known that [GK94, Theorem 2.1.11]
\[
\text{Ass}^! = \text{Ass}, \quad \text{Com}^! = \mathcal{L}ie, \quad \mathcal{L}ie^! = \text{Com}.
\]
A quadratic differential graded operad $\mathcal{P}$ is Koszul if it has the homotopy type of the differential graded dual $\mathcal{D}(\mathcal{P}^!)$ of its quadratic dual (to be discussed in section 3). The classical operads $\text{Ass}$, $\text{Com}$ and $\mathcal{L}ie$ are Koszul.

A very useful implication of this property is that for a Koszul operad $\mathcal{P}$, one can construct its canonical minimal resolution by taking the differential graded dual $\mathcal{D}$ of its quadratic dual $\mathcal{P}^!$. Algebras over the operad $\mathcal{P}_\infty := \mathcal{D}(\mathcal{P}^!)$ should be regarded as strong homotopy $\mathcal{P}$-algebras. For example, algebras over $\mathcal{D}(\text{Ass}^!)$ are precisely homotopy associative algebras ($A_\infty$-algebras). Such an algebra is a differential graded vector space $V$ with a differential $\mu_1$ of degree -1 and linear maps $\mu_k : V^\otimes k \to V$ of degree $k - 2$ defined for $k \geq 2$, subject to the relations, which, actually, come from a solution of the Maurer-Cartan equation set up in a certain differential graded Lie algebra,
\[
\sum_{\lambda=0}^{n-1} \sum_{k=1}^{n-\lambda} (-1)^{k+\lambda+k} (|v_1| + \cdots + |v_k|) \mu_{n-k+1}(v_1, \ldots, v_\lambda, \mu_k(v_\lambda+1, \ldots, v_{\lambda+k}), v_{\lambda+k+1}, \ldots, v_n) = 0
\]
holding for all $n \geq 1$ and $v_1, \ldots, v_n \in V$.

In particular, for $n = 2$ and $n = 3$, we have
\[
\mu_2(\mu_1(v_1), v_2) + \mu_2(v_1, \mu_1(v_2)) - \mu_1(\mu_2(v_1, v_2)) = 0
\]
\[
\mu_2(\mu_2(v_1, v_2), v_3) - \mu_2(v_1, \mu_2(v_2, v_3)) = \mu_1(\mu_3(v_1, v_2, v_3)) + \mu_3(\mu_1(v_1), v_2, v_3)
\]
\[
+ (-1)^{|v_1|} \mu_3(v_1, \mu_1(v_2), v_3)
\]
\[
+ (-1)^{|v_1| + |v_2|} \mu_3(v_1, v_2, \mu_1(v_3)).
\]
Here, the first identity states that the bilinear product $\mu_2$ satisfies the Leibniz rule and the second identity shows that $\mu_2$ is associative up to homotopy provided by $\mu_3$. 
The crucial property of strong homotopy $\mathcal{P}$-algebra structures is that they pass through homotopy equivalences. More precisely, the following statement holds:

**Theorem 1.30.** (Homotopy Transfer Theorem) [LV09]. Let $\mathcal{P}$ be a Koszul operad, $(V, d_V)$ and $(W, d_W)$ be differential graded $k$-vector spaces such that $V$ is a homotopy retract of $W$:

\[
\begin{array}{c}
\xymatrix{\mathcal{P} \boxtimes W \ar[r]^-p & V \\
\ar[rr]^-i \ar@{..>}[rr]^-p & & W \ar[u]^-{\text{Id}} \ar[u]^-{d_W} \ar@{..>}[rr]^-{h d_W} & & V}
\end{array}
\]

where $i$ is a quasi-isomorphism. Then any $\mathcal{P}_\infty$-structure on $W$ induces a $\mathcal{P}_\infty$-structure on $V$ in such a way that $i$ becomes a $\mathcal{P}_\infty$-quasi-isomorphism.

1.31. **Topological operads.** In this subsection we describe a few most basic operads of topological origin. Historically, those were the first elaborated examples of operads.

**Example 1.32.** A little $k$-cube is a linear embedding $c : I^k \hookrightarrow I^k$ with parallel axes. That means,

\[c(t_1, \ldots, t_n) = (c^1(t_1), \ldots, c^n(t_n)), \quad t_i \in I \text{ for } 1 \leq i \leq n,\]

where each $c^i$ is a linear function $c^i(t) = (y_i - x_i)t + x_i$ for some $0 \leq x_i < y_i \leq 1$. The little $k$-cubes operad $\mathcal{C}_k$ consists of the spaces $\mathcal{C}_k(n)$ of $n$-tuples $(c_1, \ldots, c_n)$ of little $k$-cubes such that the images of the interior of $I^k$ under all $c_i$'s are disjoint. A permutation $\sigma \in \Sigma_n$ acts on a tuple $(c_1, \ldots, c_n)$ by sending it to $(c_{\sigma(1)}, \ldots, c_{\sigma(n)})$. The composition $\gamma_{m; n_1, \ldots, n_m}(c, c_1, \ldots, c_m)$ squeezes each $c_i$ into the $i$-th slot of $c$. Below is an example of a composition computed in $\mathcal{C}_2$:

\[
\gamma\left(\begin{array}{c}
\xymatrix{1 & 2 \\
3 & 4 \\
5 & 6}
\end{array}\right) = \begin{array}{c}
\xymatrix{1 & 2 \\
3 & 4 \\
5 & 6}
\end{array}
\]

More formally, given $c : \bigsqcup_{s=1}^m I^k_s \to I^k$ and $c_s : \bigsqcup_{j=1}^{n_s} I^k_{j,s} \to I^k_s$ (where each $I^k_s$ and $I^k_{j,s}$ is just a copy of $I^k$), the composition $\gamma(c, c_1, \ldots, c_m)$ maps $\bigsqcup_{s=1}^m \bigsqcup_{j=1}^{n_s} I^k_{j,s}$ to $I^k$ by sending $I^k_{j,s}$ to $c\left(c_s\left(I^k_{j,s}\right)\right)$. The unit of $\mathcal{C}_k$ is the identical map $I^k \to I^k$. For $k \geq 2$, each $\mathcal{C}_k(n)$ is a $(k - 2)$-connected space with a free $\Sigma_n$-action.

The importance of the little cubes operad is due to the following result established by Boardman, Vogt [BV73] and May [May72]:

**Theorem 1.33.** If a based connected space $(Y, *)$ (homotopy equivalent to a CW-complex) admits the $\mathcal{C}_k$-structure, then it has the homotopy type of a based $k$-fold loop space $\Omega^k X_k$ for some space $X_k$.

The proof is done by constructing explicitly a delooping $X_k$ of $Y$ using the standard two-sided bar construction for the monad corresponding to the operad $\mathcal{C}_k$ (cf. remark 1.6).

The converse to theorem 1.33 holds as well. A based $k$-fold loop space $\Omega^k Y$ (considered as the space of maps $(I^k, \partial I^k) \to (Y, *)$) can be given the $\mathcal{C}_k$-structure by a sequence of
Theorem 1.35. A space $X$ (homotopy equivalent to a CW-complex) is homotopy equivalent to a topological monoid if and only if it is an $A_{\infty}$-space.

We will also point out an analogue of theorem 1.33:

Theorem 1.36. A based connected space $(Y, \ast)$ (homotopy equivalent to a CW-complex) has the homotopy type of a based loop space $\Omega X$ for some space $X$ if and only if $Y$ is an $A_{\infty}$-space.

1.37. Chain and homology operads. Given a topological operad $P = \{P(n)\}_{n \geq 1}$, one obtains its linearized version in the category of differential graded $k$-vector spaces (resp. graded $k$-vector spaces) by applying a chain $C_\ast$ (resp. a homology $H_\ast$) functor to each component $P(n)$. Eilenberg-Zilber and Künneth theorems imply immediately that $C_\ast(P) := \{C_\ast(P(n))\}_{n \geq 1}$ and $H_\ast(P) := \{H_\ast(P(n))\}_{n \geq 1}$ with the composition laws naturally induced from the compositions defined in the topological operad $P$ are indeed operads.

Returning to the previous example, each $K_n$ can be given a structure of a $(n-2)$-dimensional cell complex characterized by the property that

\[ \ast, \quad \theta_n : C_k(n) \times (\Omega^k Y)^n \to \Omega^k Y, \text{ where} \]

\[ \theta_n((c_1, \ldots, c_n), \gamma_1, \ldots, \gamma_n)(t) = \begin{cases} \gamma_i \circ c_i^{-1}(t), & t \in I^k \\ \ast, & \text{otherwise} \end{cases} \]

Homotopy equivalent to the little cubes operad is the little $k$-disks operad $D_k = \{D_k(n)\}_{n \geq 1}$.

For $n \geq 1$, $D_k(n)$ is the space of embeddings of $n$ labeled disjoint $k$-disks into the standard disk $D_k \subset \mathbb{R}^k$ of radius 1 centered at the origin. The composition $D \circ_i D'$ of $D \in D_k(n)$ and $D' \in D_k(m)$ is defined by cutting the $i$-th disk out of $D$ and patching the hole with the appropriately rescaled copy of $D'$.

Example 1.34. For $n \geq 2$, the associahedron $K_n$ is defined to be a convex polytope of dimension $n-2$, whose codimension $m$ faces correspond to different ways of inserting $m$ pairs of parentheses in a word of $n$ letters.

Thus, $K_2$ is a point, $K_3$ is an interval, $K_4$ is a pentagon etc. The collection $\{K_n\}_{n \geq 1}$ can be turned into a topological non-$\Sigma$ operad, called the $A_{\infty}$ operad, with the compositions

\[ c_i : K_r \times K_s \hookrightarrow K_{r+s-1} \]

being facets inclusions.

Originally, the $A_{\infty}$-operad was invented in an attempt to find the right replacement for a topological associative monoidal structure, which would be invariant with respect to homotopy equivalences. The problem here arises from the fact that on one hand, for a topological monoid $M$ and a homotopy equivalence $f : M \to X$, there is, in general, no way to equip $X$ with a monoidal structure in such a way that $f$ becomes a monoid homomorphism, even up to homotopy. On the other hand, if $X$ is a homotopy associative $H$-space, then a homotopy equivalence $g : X \to Y$ certainly induces a homotopy associative $H$-structure on $Y$, but $X$, in general, will not be homotopy equivalent to any topological monoid, that is, will not have a good “minimal model”. Thus one would like to interpolate between topological monoids, which, in a sense, seem to be too rigid, and homotopy associative $H$-spaces, which appear to be “too soft”. It turns out that the most reasonable solution to this problem is provided by the spaces that admit the action of the $A_{\infty}$-operad. We will call them $A_{\infty}$-spaces. The following is a theorem due to J.Stasheff [BV73]:

Theorem 1.35. A space $X$ (homotopy equivalent to a CW-complex) is homotopy equivalent to a topological monoid if and only if it is an $A_{\infty}$-space.

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(1) the cells of codimension \( m \) are in bijective correspondence with the ways of inserting \( m \) pairs of parentheses in a word of \( n \) letters. We denote by \( w(K) \) the expression corresponding to the cell \( K \) under this bijection;

(2) A codimension-\((m + 1)\) cell \( L \) is in the boundary of the codimension-\( m \) cell \( K \) if and only if \( w(L) \) can be obtained by inserting a pair of parentheses into \( w(K) \).

The cellular chain complexes \( \{CC_*(K_n)\}_{n \geq 1} \) form a non-\( \Sigma \) operad \( \text{Ass}_\infty \) in the category of differential graded \( k \)-vector spaces. Algebras over \( \text{Ass}_\infty \) (\( A_\infty \)-algebras, cf. remark 1.29) are differential graded algebras associative up to higher homotopies and for them a linearized version of theorem 1.35 holds:

**Theorem 1.38.** A differential graded \( k \)-vector space \( V \) is homotopy equivalent to a differential graded associative \( k \)-algebra if and only if \( V \) is an \( A_\infty \)-algebra.

**Remark 1.39.** The \( \text{Ass}_\infty \)-operad admits a more explicit combinatorial description in terms of tree complexes [Vor05] or, somewhat more conceptually, \( \text{Ass}_\infty \) can be defined as the differential graded dual of the associative operad \( \text{Ass}[GK94] \), which turns out to be its minimal model in the sense of Markl [Mar96]. This is a special case of a more general construction that produces a strongly homotopy \( \text{something} \)-algebra operad \( \mathcal{O}_\infty \) for a given \( \text{something} \)-algebra operad \( \mathcal{O} \). For instance, in a similar way, a resolution of the commutative operad \( \mathcal{Com} \) is the operad that governs \( E_\infty \)-algebras, that is, associative algebras commutative up to higher homotopies. We will discuss these constructions in more details in section 3.

### 1.40. Algebras over the homology of the little cubes operad.

For the little \( n \)-cubes operad \( \mathcal{C}_n \), a detailed description of algebras over \( H_*(\mathcal{C}_n) \) is available. This is mainly due to work of V. Arnold [Arn69] and F. Cohen [Coh76], who computed the integral homology of the configuration space of \( \text{Con}(\mathbb{R}^n, m) \) of ordered \( m \)-tuples of points in \( \mathbb{R}^n \), which is homotopy equivalent to \( \mathcal{C}_n(m) \).

**Theorem 1.41.** Let \( H_* \) be a homology functor with coefficients in a field \( k \) of characteristic zero. Then for \( n \geq 2 \), the structure of an algebra over the operad \( H_*(\mathcal{C}_n) \) is equivalent to the structure of a graded \( k \)-algebra \( A \) with

1. a graded commutative associative product \( \cdot : A \otimes A \to A \), and
2. a graded anticommutative Lie bracket \( [,] : A \otimes A \to A \) of degree \( n - 1 \)

subject to the graded Leibniz rule

\[
[a, b \cdot c] = [a, b] \cdot c + (-1)^{(|b|+n)-|c|} [a, c] \cdot b
\]

In particular, due to the fact that for any space \( X \) and \( n \geq 1 \), the loop space \( \Omega^{n+1}X \) has a structure of a \( \mathcal{C}_{n+1} \)-space, the homology of an iterated loop space enjoys having the above operations. Moreover, in the same work [Coh76] F. Cohen shows how the action of \( H_*(\mathcal{C}_{n+1}) \) on \( H_*(X) \) gives rise to Kudo-Araki operations, introduced in [KA56],

\[
Q^i_{(2)} : H_p(X;\mathbb{Z}_2) \to H_{p+i}(X;\mathbb{Z}_2), \quad \text{for } p \leq i \leq p+n,
\]

Dyer-Lashof operations introduced in [DL62]

\[
Q^{i}_{(r)} : H_p(X;\mathbb{Z}_r) \to H_{p+2i(r-1)}(X;\mathbb{Z}_r), \quad \text{for } p(r-1) \leq 2i(r-1) < p(r-1)+n,
\]

and Browder operations [Bro60]

\[
\lambda_n : H_p(X) \otimes H_q(X) \to H_{p+q+n}(X)
\]

In the special case, when \( n = 2 \), algebras over \( H_*(\mathcal{C}_2; k) \) with \( k \) of characteristic zero are known under the name of Gerstenhaber algebras. Below we list a few basic examples of such algebras.
Example 1.42.

(1) **Exterior algebra of a Lie algebra.** For a finite dimensional Lie algebra \( \mathfrak{g} \), the exterior algebra \( \Lambda^\bullet(\mathfrak{g}) \) is a Gerstenhaber algebra. Here, the graded commutative product is the exterior product and the Gerstenhaber bracket is a natural extension of the original Lie bracket.

(2) **Algebra of multivector fields.** Let \( X \) be a smooth manifold. A multivector field is an element of the exterior algebra bundle \( \Lambda^\bullet_{\infty}(X) (\Gamma(TX)) \). As in the previous example, the graded commutative product is taken to be the exterior product and the bracket is the Schouten-Nijenhuis operation. It is defined in terms of the usual Lie bracket of vector fields as

\[
[a_1 \ldots a_m, b_1 \ldots b_n]_{SN} := \sum_{i,j} (-1)^{i+j} [a_i, b_j] a_1 \ldots a_{i-1}a_{i+1} \ldots a_n b_1 \ldots b_j b_{j+1} \ldots b_n
\]

for vector fields \( a_i, b_j \).

(3) **Hochschild cohomology.** The standard cup product \( CH^p(A; A) \otimes CH^q(A; A) \to CH^{p+q}(A; A) \) on the Hochschild cochains of an associative \( k \)-algebra is given by the formula

\[
(f \smile g)(a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q) := f(a_1 \otimes \cdots \otimes a_p) g(b_1 \otimes \cdots b_q).
\]

It is evidently associative, but (graded) commutativity holds only up to a chain homotopy, as it was first observed by M. Gerstenhaber [ger1]. Namely, for \( f \in CH^p(A; A), g \in CH^q(A; A) \), the following identity holds

\[
g \smile f - (-1)^{pq} f \smile g = (-1)^{q} f \circ \delta g - (-1)^{q} \delta(f \circ g) + \delta f \circ g,
\]

where \( \delta \) is the Hochschild differential and \( \circ : CH^p(A; A) \otimes CH^q(A; A) \to CH^{p+q-1}(A; A) \) is defined by

\[
(f \circ g)(a_1 \otimes \cdots a_{p+q-1}) := \sum_{i=1}^{p} (-1)^{(i-1)q} f(a_1 \otimes \cdots a_{i-1} \otimes g(a_i \otimes \cdots a_{i+q-1}) \otimes a_{i+q} \otimes \cdots a_{p+q-1}).
\]

Now, defining

\[
[f, g] := f \circ g - (-1)^{(p-1)(q-1)} g \circ f
\]

for \( f \in CH^p(A; A) \) and \( g \in CH^q(A; A) \), and passing to cohomology, gives a Gerstenhaber algebra structure on \( HH^\bullet(A; A) \).

**Remark 1.43.** The fact that a Gerstenhaber algebra structure is present on both homology of \( \mathcal{C}_2 \)-spaces and Hochschild cohomology of associative algebras led P. Deligne [Del90] to conjecture that the action of \( H_*(\mathcal{C}_2) \) on \( HH^\bullet(A; A) \) lifts up to the chain level. That is, there exists an operad (in \( \text{dgVect}_k \)) acting on \( CH^\bullet(A; A) \), which is chain homotopy equivalent to the operad \( \mathcal{C}_* \). Presence of this operad action would imply existence of operations in Hochschild cohomology similar to homology operations for double loop spaces listed above. For homology with coefficients in a field of characteristic zero the statement was proved by M. Kontsevich [Kon99], D. Tamarkin [Tam98], A. Voronov [Vor05]. Over \( \mathbb{Z} \) the result is due to J. McClure, J. H. Smith [MS02], M. Kontsevich and Y. Soibelman [KS00], Berger and Fresse [BF04]. Explicit formulas for Dyer-Lashof operations in Hochschild cohomology can be found in a work of V. Tourchine [Tou06].

Later, in section 5, we will work with a certain subclass of Gerstenhaber algebras, known as Batalin-Vilkovisky algebras.
1.44. **Geometric operads.** The following examples will serve as prototypes for the *cyclic* and *modular* operads, which we will introduce in the next section.

**Example 1.45.** A holomorphic disk on a Riemann sphere $\mathbb{C}P^1$ is a point $p \in \mathbb{C}P^1$ (which we call a puncture) together with a holomorphic embedding of the standard closed disk $U = \{z \in \mathbb{C} | |z| \leq 1\}$ to $\mathbb{C}P^1$ centered at $p$. Let $\mathcal{S}(n)$ be the set of Riemann spheres with $n + 1$ disjoint labeled holomorphic disks $u_0, u_1, \ldots, u_n$, modulo the action of $PSL(2, \mathbb{C})$. The disk labeled with zero will be called the output and all other disks will be referred to as inputs. Relabeling the disks $u_1, \ldots, u_n$ gives an action of $\Sigma_n$ on $\mathcal{S}(n)$ and the operation

$$\circ_i : \mathcal{S}(n) \times \mathcal{S}(m) \to \mathcal{S}(n + m - 1)$$

$$(M, N) \mapsto M \circ_i N$$

that sews the output of $M$ to the $i$-th input of $N$ by cutting out small disks around the punctures and then identifying the annuli via $z^M_0 z^N_1 = 1$ in some local chart, yields a composition law that turns $\{\mathcal{S}(n)\}_{n \geq 1}$ into an operad.

By the results of Y.-Z. Huang [Hua97], an algebra over the operad $\mathcal{S}$ constitutes the same data as a conformal field theory in genus 0.

**Example 1.46.** Let $M_{0,n+1}$ be the moduli space of $(n + 1)$-tuples of distinct points on the complex projective line $\mathbb{C}P^1$ modulo projective automorphisms. The space $M_{0,n+1}$ has a canonical compactification $\overline{M}(n)$, which is a smooth projective complex variety of dimension $n - 2$, introduced by Grothendieck and Knudsen [Del72], [Knu83]. Specifically, the space $\overline{M}(n)$ is the moduli space of stable $(n+1)$-pointed genus zero curves. Such a curve $(C; x_0, \ldots, x_n)$ is, by definition, a possibly reducible curve with at most nodal singularities and distinct points $x_0, \ldots, x_n \in C$ such that

1. each component of $C$ is isomorphic to $\mathbb{C}P^1$;
2. the graph of intersection of components of $C$ is a tree;
3. each component of $C$ has at least three special points, where a special point means either a singular point of $C$ or one of $x_i$’s.

We turn the collection of spaces $\overline{M} := \{\overline{M}(n)\}_{n \geq 1}$ into a non-unital topological operad, which we will call the *configuration operad*, as follows. For $n \geq 1$, $\sigma \in \Sigma_n$ acts on $\overline{M}(n)$ by

$$(C; x_0, \ldots, x_n) \mapsto (C; x_0, x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

and the composition map is defined by

$$\gamma : \overline{M}(k) \times \overline{M}(n_1) \times \ldots \overline{M}(n_k) \to \overline{M}(n_1 + \cdots + n_k)$$

$$(\{(C_1; y_1, \ldots, y_k), (C_1; x_1^{(1)}, \ldots, x_1^{(n_1)}), \ldots, (C_k; x_k^{(1)}, \ldots, x_k^{(n_k)})\}) \mapsto (C'; y_0, x_1^{(1)}, \ldots, x_{n_1}^{(1)}, \ldots, x_k^{(k)}),$$

where $C'$ is obtained from $C \coprod C_1 \cdots \coprod C_k$ by identifying each $y_i$ with $x_0^i$ for $i = 1, \ldots, k$. 
2. Cyclic and modular operads

One observes that the punctured Riemann spheres operad $S$ described in example 10 of the previous section admits a somewhat higher symmetry than it was originally given. Namely, the action of $\Sigma_n$ on the labels of inputs $\{1, \ldots, n\}$ can be extended to the action of $\Sigma_n^+ \simeq \Sigma_{n+1}$ on all the labels $\{0, 1, \ldots, n\}$, thus allowing permutations of input disks with the output. A similar situation takes place for the configuration operad $M(n)$, where the action of the symmetric group $\Sigma_n$ on marked points $x_1, \ldots, x_n$ of a stable curve $(C; x_0, x_1, \ldots, x_n)$ can be extended to the action of $\Sigma_{n+1}$ on all $n+1$ marked points. The idea of an operad with some extra symmetry allowing interchanging “inputs” of the “operations” with their “outputs” was made precise by E.Getzler and M.Kapranov in [GK95] and found its realization in the notions of cyclic and modular operads.

Definition 2.1. Denote the cycle $(0\ 1\ \ldots\ n)$ in $\Sigma_n^+ \simeq \Sigma_{n+1}$ by $\tau_n$. An operad $\mathcal{P}$ in a strict symmetric monoidal category $(C, \otimes, I)$ is called cyclic if the action of $\Sigma_n$ on $\mathcal{P}(n)$ extends to an action of $\Sigma_n^+$ and the extended action satisfies the following conditions:

(a) if $\eta \in \mathcal{P}(1)$ is the unit of $\mathcal{P}$, then $\tau_1(\eta) = \eta$.

(b) For any $m, n \geq 1$, the diagram

$$
\begin{array}{c}
\mathcal{P}(m) \otimes \mathcal{P}(n) \\
\tau_m \otimes \tau_n \\
\mathcal{P}(m) \otimes \mathcal{P}(n) \\
\text{swap} \\
\mathcal{P}(n) \otimes \mathcal{P}(m) \\
\end{array}
\xrightarrow{\circ_{m+n-1}}
\begin{array}{c}
\mathcal{P}(m+n-1) \\
\end{array}
$$

commutes.

(c) For any $m, n \geq 1$ and $2 \leq i \leq m$, the diagram

$$
\begin{array}{c}
\mathcal{P}(m) \otimes \mathcal{P}(n) \\
\tau_m \otimes \text{id} \\
\mathcal{P}(m) \otimes \mathcal{P}(n) \\
\end{array}
\xrightarrow{\circ_{i-1}}
\begin{array}{c}
\mathcal{P}(m+n-1) \\
\end{array}
$$

commutes.

Here, $\tau_n$ acts on an operation $p \in \mathcal{P}(n)$ by making the output of $p$ to be the $n$-th input of $\tau_n \cdot p$ and the first input of $p$ to be the output of $\tau_n \cdot p$:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{cyclic_operad.png}
\caption{Cyclic operad action.}
\end{figure}

Remark 2.2. An alternative set of axioms characterizing cyclic operads can be given in terms of the compositions

$$
i \circ j : \mathcal{P}(m) \otimes \mathcal{P}(n) \rightarrow \mathcal{P}(m+n-1), \quad 0 \leq i \leq m, \ 0 \leq j \leq n.$$

Instead of stating these axioms explicitly, we confine ourselves to giving a pictorial description of such composition laws and the action of $\Sigma_n^+$. 

Remark 2.3. In the same way as operads can be defined to be algebras over the monad of rooted oriented trees $T$, cyclic operads can be characterized as algebras over the monad of unrooted trees $T^+$ in $\text{Mod}_{C-\Sigma}$. The construction is analogous to the one presented in subsection 1.13. Namely, for a $\Sigma$-module $E$, $n \geq 1$ and a finite unrooted non-oriented tree $T$ with $n+1$ leaves, we set

$$E((T)) := \bigotimes_{v \in \text{Vert}(T)} E(\text{Leg}(v)),$$

and then build up the $\Sigma$-module $T^+_E = \{T^+_E(n)\}_{n \geq 1}$ by taking

$$T^+_E(n) := \text{colim}_{T \in \text{uTree}_{n+1}} E((T)),$$

where $\text{uTree}_{n+1}$ is the category of finite unrooted non-oriented trees with $n+1$ leaves and their isomorphisms. The double parentheses around $T$ are meant to emphasize that $T$ is acted on by the extended symmetric group $\Sigma^+_{n+1}$ rather than just $\Sigma_n$. The monadic product $T^+_2E \to T^+_1E$ is induced by the grafting of trees and the unit $\text{Id} \to T^+_0$ morphism sends the $n$-th component $E(n)$ of a $\Sigma$-module $E$ to $E((\ast_{n+1}))$, where $\ast_{n+1}$ is the tree with $n+1$ leaves and no internal edges.

2.4. Cyclic endomorphism operad. The following construction is the cyclic version of the endomorphism operad defined in the previous section.

Example 2.5. Let $V$ be a finite-dimensional $k$-vector space and $B : V \otimes V \to k$ be a non-degenerate symmetric bilinear form. The form $B$ induces an isomorphism

$$\hat{B} : \text{Hom}(V^{\otimes n}, V) \to \text{Hom}(V^{\otimes (n+1)}, k)$$

$$f \mapsto B(\cdot, f(\cdot)).$$

For any $n \geq 1$, the standard $\Sigma^+_{n}$-action on $\text{Hom}(V^{\otimes (n+1)}, k)$ defined by

$$(f \cdot \sigma)(v_0, \ldots, v_n) := f(v_{\sigma^{-1}(0)}, \ldots, v_{\sigma^{-1}(n)})$$

induces via $\hat{B}$ a $\Sigma^+_{n}$-action on $\text{Hom}(V^{\otimes n}, V)$, which is the degree $n$ component of the standard endomorphism operad $\text{End}_V$. Thus $\{\text{End}_V(n)\}_{n \geq 1}$ with the $\Sigma^+_{n}$-action forms a cyclic operad. We would refer to it as cyclic endomorphism operad of the pair $(V, B)$.

Example 2.6. The operads $\text{Com}, \text{Ass}$, the operad of punctured Riemann spheres $S$ and the configuration operad $\overline{M}$ defined in the previous section are cyclic. The homology operad of little 2-cubes $H_*(\mathbb{C}_2)$ is not.
2.7. **Algebras over cyclic operads.** As before, the main point of our interest is not just the cyclic operads themselves, but rather objects endowed with an action of such an operad. To make sense of it, we need the following definition.

**Definition 2.8.** Let \( \mathcal{C} \) be a cyclic operad in \( \text{Vect}_k \), \( V \) be a \( k \)-vector space and \( B : V \otimes V \to k \) be a non-degenerate symmetric bilinear form. Then \( V \) is called an algebra over a cyclic operad \( \mathcal{C} \) if there is a morphism of cyclic operads \( \mathcal{C} \to \text{End}_V \).

In a similar fashion, one can define algebras over cyclic operads in the categories of graded or differential graded \( k \)-vector spaces.

**Example 2.9.** An algebra over a cyclic operad \( \text{Com} \) is a commutative Frobenius algebra, that is, a \( k \)-vector space \( V \) equipped with a symmetric bilinear non-degenerate form \( B : V \otimes V \to k \) and a associative multiplication \( \cdot : V \otimes V \to V \) subject to the condition

\[
B(a \cdot b, c) = B(a, b \cdot c).
\]

**Remark 2.10.** For algebras over a cyclic operad, one can define cyclic cohomology generalizing cyclic cohomology for associative algebras as the non-abelian derived functor of the universal bilinear form \([\text{GK}95]\).

2.11. **Modular operads.** The idea of a modular operad is motivated by the following observation. Given a punctured Riemann sphere \( S \) with numbered holes, then sewing the \( i \)-th hole of \( S \) to its \( j \)-th hole (as described in example 1.) would result in a new genus 1 surface with \( n - 2 \) holes. More generally, consider the collection \( M := \{M(g, n)\}_{g \geq 0, n \geq 0} \), where \( M(g, n) \) is the moduli space of genus \( g \) Riemann surfaces with \( n \) holes. Sewing the \( i \)-th hole of a surface \( S \in M(g, m) \) to the \( j \)-th hole of \( P \in M(h, n) \) yields a surface \( S_i \cup_j P \in M(g + h, m + n - 1) \) and sewing the \( i \)-th hole of \( S \) to its \( j \)-th hole produces the surface \( \xi_{ij}(S) \in M(g + 1, m - 2) \). A modular operad is a formalization of this construction.

Before we give its formal definition, we recite the following heuristic slogan:

*Operads are related to rooted trees in the same way as cyclic operads are related to unrooted trees and modular operads are related to graphs.*

From now on, we will be working in the category \( \text{dgVect}_k \) of differential graded \( k \)-vector spaces.

**Definition 2.12.** A modular \( \Sigma \)-module is a sequence \( \mathcal{E} = \{\mathcal{E}(g, n)\}_{g \geq 0, n \geq 0} \) of differential graded \( k \)-vector spaces such that each \( \mathcal{E}(g, n) \) is equipped with a \( \Sigma_n \)-action. A morphism \( \mathcal{E} \to \mathcal{F} \) of modular \( \Sigma \)-modules is a collection of equivariant maps \( \mathcal{E}(g, n) \to \mathcal{F}(g, n) \) defined for all \( g \) and \( n \).

We say that a modular \( \Sigma \)-module \( \mathcal{E} \) is stable if \( \mathcal{E}(g, n) = 0 \) whenever \( 2g - 2 + n \leq 0 \).

Sometimes we will need the components of a modular \( \Sigma \)-module \( \mathcal{E} \) be indexed by elements of an arbitrary finite set, rather than just non-negative integers. To this end, for \( g \geq 0 \) and arbitrary finite set \( I \) of cardinality \( n \), we define

\[
\mathcal{E}(g, I) := \left( \bigoplus_{\text{Iso}(n, I)} \mathcal{E}(g, n) \right)_{\Sigma_n}
\]

Then \( \mathcal{E}(g, I) \) is naturally equipped with a (right) action of \( \text{Aut}(I) \).

**Remark 2.13.** The terminology and the inequality in the definition of stable \( \Sigma \)-modules come from the moduli theory of curves (cf. example 1.46), where the inequality \( 2g - 2 + n > 0 \) guarantees that the automorphism group of a genus \( g \) curve with \( n \) marked points is finite. Similarly, in our case, the stability condition is needed to gain a certain finiteness property (cf. lemma 2.18).
For a stable modular $\Sigma$-module $A = \{A(g,n)\}_{g \geq 0, n \geq 0}$, we define a $\Sigma$-module $\{A(n)\}_{n \geq 0}$ by taking

$$A(n) := \bigoplus_{g \geq 0} A(g,n).$$

**Definition 2.14.** A **modular operad** structure on $A = \{A(g,n)\}_{g \geq 0, n \geq 0}$ consists of the following data:

1. cyclic operad structure on the $\Sigma$-module $\{A(n)\}_{n \geq 0}$;
2. a family of **contraction maps** $\xi_{ij}$ defined for all $g \geq 0$, all finite sets $I$ and distinct $i, j \in I$. Namely, contractions maps are morphisms

$$\xi_{ij} : A(g,I) \rightarrow A(g+1,I - \{i,j\})$$

subject to the following conditions:

(a) **Equivariance:** for any $\sigma \in \text{Aut}(I)$ such that $\sigma(\{i,j\}) = \{i,j\}$ and any $a \in A(g,I)$,

$$\xi_{\sigma(i)\sigma(j)}(a \cdot \sigma_{ij}) = \xi_{ij}(a \cdot \sigma),$$

where $\sigma_{ij}$ is the restriction of $\sigma$ to $I - \{i,j\}$.

(b) **Commutativity:** for distinct $i, j, k, l \in I$, $\xi_{ij} \circ \xi_{kl} = \xi_{kl} \circ \xi_{ij}$.

(c) **Compatibility with the cyclic operad structure:** let $S, T$ be distinct finite sets, $x \in S, y \in T$ and $i,j \in S \bigcup T \setminus \{x,y\}$, then

$$\xi_{ij}(a \circ_y b) = \begin{cases} 
\xi_{xy}(a_i \circ_j b), & i \in S, j \in T \\
\xi_{ij}(a) \circ_y b, & i, j \in S \\
a \circ_x \xi_{ij}(a), & i, j \in T
\end{cases}$$

for any $a \in A(g,S), b \in A(g,T)$.

The meaning of these compatibility conditions can be explained pictorially:

$$\xi_{ij} \left( \begin{array}{c}
\circ_i \\
\circ_j \\
\bullet
\end{array} \right) = \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array} = \xi_{ij} \left( \begin{array}{c}
\circ_i \\
\circ_j \\
\bullet
\end{array} \right)$$

$$\xi_{ij} \left( \begin{array}{c}
\circ_i \\
\circ_j \\
\bullet
\end{array} \right) = \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array} = \xi_{ij} \left( \begin{array}{c}
\circ_i \\
\circ_j \\
\bullet
\end{array} \right)$$

2.15. **The monad of stable graphs.** In this subsection we describe the monad of stable graphs $M$, which is an analogue of the monads of rooted and unrooted trees defined in 1.16 and 2.3. Modular operads will be characterized then as algebras over $M$.

Thinking of our construction as of a higher genus generalization of the monad of unrooted trees, at the first step we replace the groupoid of unrooted trees by the groupoid of stable labeled graphs.

**Definition 2.16.** A **labeled graph** $\Gamma$ is a connected graph with a mapping $g : \text{Vert}(\Gamma) \rightarrow \mathbb{Z}_{\geq 0}$.

The **genus** of a labeled graph $\Gamma$ is

$$g(\Gamma) := b_1(\Gamma) + \sum_{v \in \text{Vert}(\Gamma)} g(v).$$
A morphism of labeled graphs \( f : \Gamma_0 \to \Gamma_1 \) is a morphism of underlying graphs such that for any vertex \( v \) of graph \( \Gamma_1 \), the genus of the graph \( f^{-1}(v) \) is equal to \( g(v) \).

**Definition 2.17.** A labeled graph \( \Gamma \) is called stable if for any vertex \( v \in \text{Vert}(\Gamma) \),
\[
2(g(v) - 1) + |\text{Leg}(v)| > 0.
\]

**Lemma 2.18.** There are finitely many isomorphism classes of stable graphs with the prescribed genus and number of legs.

The statement can be proved by, first, establishing the identity
\[
3(g(\Gamma) - 1) + |\text{Leg}(\Gamma)| = |\text{Edge}(\Gamma)| + \sum_{v \in \text{Vert}(\Gamma)} \left(3(g(v) - 1) + |\text{Leg}(v)|\right)
\]
via a simple counting argument, and then noticing that the stability condition gives the upper bound of \( 3(g(\Gamma) - 1) + |\text{Leg}(\Gamma)| \) for \( |\text{Edge}(\Gamma)| \). The presence of an upper bound for the number of edges implies that, up to isomorphism, there are finitely many stable graphs \( \Gamma \) with the fixed values of \( g(\Gamma) \) and \( |\text{Leg}(\Gamma)| \).

**Definition 2.19.** For a finite set \( S \) and non-negative integer \( g \), let \( \Gamma(g, S) \) be the category of pairs \((\Gamma, \rho)\), where \( \Gamma \) is a stable labeled graph of genus \( g \) and \( \rho \) is a bijection \( \text{Leg}(\Gamma) \to S \).

Morphisms in \( \Gamma(g, S) \) are all labeled graph isomorphisms preserving the labelling \( \rho \) of the legs. By \( \text{Iso} \Gamma(g, S) \) we denote the subcategory of all isomorphisms of \( \Gamma(g, S) \).

For a stable graph \( \Gamma \) and a stable \( \Sigma \)-module \( E \), we let
\[
E(\Gamma) := \bigotimes_{v \in \text{Vert}(\Gamma)} E(g(v), \text{Leg}(v)),
\]
and define the endofunctor \( M \) on the category of stable modular \( \Sigma \)-modules by the formula
\[
M E(g, S) := \operatorname{colim}_{\Gamma \in \text{Iso} \Gamma(g, S)} E(\Gamma).
\]

Since \( \text{Iso} \Gamma(g, S) \) is a groupoid there is a non-canonical isomorphism
\[
M E(g, S) \simeq \bigoplus_{\Gamma \in \{\text{Iso} \Gamma(g, S)\}} E(\Gamma)_{\text{Aut}(\Gamma)},
\]
where the direct sum is taken over the set of representatives of the isomorphism classes of stable genus \( g \) graphs with the legs labelled by the elements of \( S \). Lemma 2.18 guarantees that this direct sum is finite.

Now, we are going to put a monad structure on \( M \). For a category \( C \) and \( k \geq 0 \), denote by \( \text{Iso}_k C \) the category of diagrams
\[
X_0 \to X_1 \to \cdots \to X_k \quad (X_i \in C)
\]
and their isomorphisms. By induction one proves the following formula for iterated compositions of \( M \).

**Lemma 2.20.** For \( k \geq 0 \),
\[
(M^{k+1} E)(g, S) \simeq \operatorname{colim}_{\Gamma_0 \to \cdots \to \Gamma_n} \operatorname{colim}_{\Gamma_0 \in \text{Iso} \Gamma(g, S)} E(\Gamma_0)\]
As a consequence, for any \( g \) and \( S \), there is a natural morphism
\[
M^2 E(g, S) \simeq \operatorname{colim}_{\Gamma_0 \to \Gamma_1} \operatorname{colim}_{\Gamma_0 \in \text{Iso} \Gamma(g, S)} E(\Gamma_0) \to \operatorname{colim}_{\Gamma_0 \in \text{Iso} \Gamma(g, S)} E(\Gamma_0) \simeq ME(g, S)
\]
induced by the functor

$$\text{Iso}_1 \Gamma(g, S) \to \text{Iso}_0 \Gamma(g, S)$$

$$\left[ \Gamma_0 \to \Gamma_1 \right] \mapsto \left[ \Gamma_0 \right].$$

That gives a natural transformation $\mu : M^2 \to M$. The unit $\nu : \text{Id} \to M$ is defined by including $\mathcal{E}(\ast_g S)$ as a summand into (5), where $\ast_g S$ is the graph with a single vertex of genus $g$ and $|S|$ half-edges labelled by $S$. The transformations $\mu$ and $\nu$ form a monad $(M, \mu, \nu)$ in the category of stable modular $\Sigma$-modules.

**Theorem 2.21.** ([MSS07] Theorem 5.41) **Modular operads are algebras over the monad $(M, \mu, \nu)$.**

**Remark 2.22.** For a stable modular $\Sigma$-module $\mathcal{E}$, $\mathcal{M} \mathcal{E}$ is the $\Sigma$-module underlying the free modular operad generated by $\mathcal{E}$.

**Example 2.23.** Any cyclic operad $M = \{M(n)\}_{n \geq 1}$ automatically gives rise to a modular operad $\hat{M}$ by setting

$$\hat{M}(g, n) := \begin{cases} M(n), & g = 0 \\ 0, & g > 0 \end{cases}.$$

Again, this reflects our paradigm that cyclic operads comprise the genus zero case of modular operads.

**Example 2.24.** Let $(V, B)$ be a differential graded $k$-vector space with a non-degenerate graded symmetric bilinear form $B : V \otimes V \to k$. Encouraged by theorem 2.21, we will define the modular *endomorphism operad* $\mathcal{E}nd_V$ as a certain algebra over $M$. By (5), this amounts to constructing morphisms

$$\alpha_\Gamma : \mathcal{E}nd_V(\Gamma) \to \mathcal{E}nd_V(g, S)$$

for each $g \geq 0$, finite set $S$ and $\Gamma \in \text{Iso}\Gamma(g, S)$ in the way compatible with the product structure on $M$.

First, for $g \geq 0$ and a finite set $S$ of cardinality $n$, we take $\mathcal{E}nd(g, S) := V^\otimes S$, where

$$V^\otimes S := \left( \bigotimes_{\text{Iso}(n, S)} V \right)_{\Sigma_n}.$$

Then, by (4), for a graph $\Gamma \in \Gamma(g, S)$, we have $\mathcal{E}nd(\Gamma) = V^{\text{Flag}(\Gamma)}$.

Now, we extend $B$ to the multilinear form

$$B^{\text{Edge}(\Gamma)} : V^{\text{Flag}(\Gamma)} \to V^{\text{Leg}(\Gamma)},$$

which contracts (using $B$) the factors of $V^{\text{Flag}(\Gamma)}$ corresponding to the flags that pair up to an edge. For example,

$$B^{\text{Edge}(\Gamma)}(v_1 \otimes v_2 \otimes \cdots \otimes v_9) = B(v_3, v_5)B(v_4, v_6)B(v_7, v_8)v_1 \otimes v_2 \otimes v_9,$$

where $\Gamma$ is the graph shown below.

![Graph](image-url)
Finally, we define \( \alpha_\Gamma \) to be
\[
\alpha_\Gamma : \text{End}_V(\Gamma) \simeq V^{\text{Flag}(\Gamma)} \xrightarrow{B^{\text{Edge}(\Gamma)}} V^{\text{Leg}(\Gamma)} \simeq V \otimes S = \text{End}(g, S).
\]

**Definition 2.25.** An algebra \((V, B)\) over a modular operad \(A\) is given by a graded \(k\)-vector space \(V\) with a non-degenerate graded symmetric bilinear form \(B : V \otimes V \to k\) and a morphism of modular operads \(A \to \text{End}_V\).

**Remark 2.26.** The definition of \(B^{\text{Edge}(\Gamma)}\) in example 2.24 resembles a certain part of the Feynman diagram expansion formula, which is used in physics for writing out perturbative series of correlation functions as formal sums over graphs. Since another instance of this formula will also show through some constructions that we are about to discuss in section 4, we provide a little more details on the Feynman diagram calculus in the appendix.

### 3. Cobar resolution of a dg-operad and homotopy algebras

In this section, we discuss in more details the notion of a homotopy algebra over an operad in the differential graded context.

According to the general principle [BM03, Corollary 4.5], given an operad \(P\) in a closed symmetric monoidal model category \(C\) satisfying some additional (rather mild) technical assumptions, strong homotopy \(P\)-algebras should be defined as algebras over a cofibrant resolution \(P_\infty\) of \(P\), where cofibrancy is understood in the sense of a certain model structure on the category of operads in \(C\), which is induced from the model structure on \(C\). This principle is justified by the following theorem, which states that the \(P_\infty\)-algebra structure indeed enjoys being homotopy invariant.

**Theorem 3.1.** ([BM03] Theorem 3.5) Let \(f : X \to Y\) be a morphism in a monoidal model category in which the operads carry a transferred model structure; let \(\tilde{P}\) be a cofibrant operad.

1. If \(Y\) is fibrant and \(f^{\otimes n}\) is a trivial cofibration for all \(n \geq 1\), then a \(\tilde{P}\)-algebra structure on \(X\) extends along \(f\) to a \(\tilde{P}\)-algebra structure on \(Y\).
2. If \(X\) is cofibrant and \(f\) is a trivial fibration, then a \(\tilde{P}\)-algebra structure on \(Y\) lifts up along \(f\) to a \(\tilde{P}\)-algebra structure on \(X\).
3. If \(X\) and \(Y\) are cofibrant-fibrant and \(f\) is a weak equivalence, then any \(\tilde{P}\)-algebra structure on \(X\) induces a \(\tilde{P}\)-algebra structure on \(Y\) and vice versa. The induced structure is compatible with \(f\), meaning that \(f\) preserves the \(\tilde{P}\)-algebra structure up to homotopy.

A cofibrant resolution for a topological operad is provided by the classical \(W\)-construction of Boardman and Vogt [BV73]. For a differential graded operad \(P\), a cofibrant resolution can be built up in the form of the cobar resolution \(D(D(P)) \xrightarrow{\tau} P\), where \(D(P)\) is the differential-graded dual of \(P\). This is analogous to the well-known cobar construction of F. Adams [Ada56] for differential graded algebras.

We will briefly describe both of these constructions in the following two subsections, although an account of the Boardman-Vogt resolution would be given just for illustrative purposes and, unlike the cobar construction, it will no longer be used in the paper.

The upshot of this section is a theorem, which states that the data defining an algebra over the dg-dual of a dg-operad \(P\) is encoded in a solution of the Maurer-Cartan equation set up in a certain differential graded algebra. This result can be regarded as a precursor or the “genus zero” case of Barannikov’s theorem.
3.2. Boardman-Vogt resolution of a topological operad. The idea behind the following construction is to modify the free operad definition given in 1.13 by assigning lengths to the edges of the trees in $\text{Tree}_n$ ($n \geq 1$).

We say that an oriented tree is reduced if it has no vertices with exactly one incoming flag.

**Definition 3.3.** A metric tree is a pair $(T, h_T)$, where $T$ is a rooted oriented reduced tree and $h_T$ is a function $\text{Edge}(T) \to [0; 1]$. The set of all metrics on a tree $T$ will be denoted by $\text{Met}(T)$. Clearly, $\text{Met}(T)$ can be (non-canonically) identified with $I^k$, where $k = |\text{Edge}(T)|$.

For a topological operad $\mathcal{P}$ with $\mathcal{P}(1) = *$ (the one-point space) we consider a non-unital operad $\mathcal{P}^+$ obtained from $\mathcal{P}$ by setting $\mathcal{P}(1) := \emptyset$. As in 1.13, for a rooted oriented tree $T$ we set

$$\mathcal{P}^+(T) := \times_{v \in \text{Vert}(T)} \mathcal{P}(\text{In}(v)).$$

Observe that if $T/e$ is the tree obtained from $T$ by contracting the edge $e$, then the graph morphism $T \to T/e$ induces

1. a natural morphism $\gamma_e : \mathcal{P}^+(T) \to \mathcal{P}^+(T/e)$ coming from the operad structure on $\mathcal{P}^+$,
2. an inclusion $s_e : \text{Met}(T/e) \to \text{Met}(T)$ which extends a metric $h_{T/e}$ to $h_T$ by setting $h_T(e) = 0$.

Now, we define a non-unital topological operad $W\mathcal{P}^+$ as follows. For $n \geq 2$,

$$W\mathcal{P}^+(n) := \coprod_{T \in \{\text{rTree}_n\}} \text{Met}(T) \times \mathcal{P}^+(T)/\sim,$$

where the coproduct is taken over the set of isomorphism classes of rooted reduced oriented trees with $n$ leaves and the equivalence relation is

$$(s_e(h_T), f) \sim (h_T, \gamma_e(f)).$$

Intuitively, this means that a metric tree $T$ with an edge $e$ of “zero length” ($h_T(e) = 0$) is identified with the tree $T/e$, where the edge $e$ is contracted.

The composition operations on $W\mathcal{P}^+$ are induced by grafting metric trees, and each $W\mathcal{P}^+(n)$ is taken with the quotient topology. Finally, by adjoining the unit element to $W\mathcal{P}^+$ we obtain a topological operad $W\mathcal{P}$ with the augmentation map $W\mathcal{P} \to \mathcal{P}$ defined for each $n \geq 2$ and $T \in \{\text{rTree}_n\}$, by

$$\text{Met}(T) \times \mathcal{P}^+(T) \xrightarrow{\text{proj}} \mathcal{P}^+(T) \xrightarrow{\gamma_T} \mathcal{P}(n)$$

where $\gamma_T$ composes along the edges (using the operad structure of $\mathcal{P}$) all the operations assigned to the vertices of $T$ by (6).

The operad $W\mathcal{P}$ is called the Boardman-Vogt resolution of $\mathcal{P}$ or the $W$-construction on $\mathcal{P}$.

**Theorem 3.4.** ([BV73] Proposition 2.3.6, Theorem 2.3.17) The operad $W\mathcal{P}$ is cofibrant with respect to the model structure induced on the category of topological operads from the standard model structure on the category of compactly generated topological spaces. The augmentation map $W\mathcal{P} \to \mathcal{P}$ is a weak equivalence.

**Example 3.5.** For the topological monoid operad $\text{Mon} = \{\text{Mon}(n)\}_{n \geq 1}$, where $\text{Mon}(n) = \Sigma_n$ with the discrete topology, $W\text{Mon}(n)$ is homotopy equivalent to the $n$-th Stasheff associahedron $K_n$ (cf. example 1.33).
Remark 3.6. There is a more general version of the $W$-construction due to C.Berger and I.Moerdijk [BM06], which works in a monoidal model category $C$ with a segment object $H$. The latter is an object of $C$ with the morphisms $\vee: H \otimes H \to H$, $0: I \to H$, $1: I \to H$ subject to conditions
\[
x \vee (y \vee z) = (x \vee y) \vee z, \quad 0 \vee x = x \vee 0, \quad 1 \vee x = 1 = x \vee 1.
\]
One can think of $H$ as of a unit interval $[0;1]$ with an operation $\vee = \max(-,-)$. A slight modification of the topological construction described above yields for an operad $P$ in $C$, an operad $W(H,P)$ with an augmentation $W(H,P) \to P$. Under certain additional technical assumptions on $C$ and $P$, it can be shown that there exists a cofibration followed by a weak equivalence
\[
L \circ U(P) \to W(H,P) \to P.
\]
Here, $L \circ U$ produces a free operad generated by $\{P(n)\}_{n \geq 1}$ considered as a $\Sigma$-module.

3.7. **Cobar resolution of a differential graded operad.** In what follows, for a differential graded $k$-vector space $V$, we denote by $V^*$ its dual defined, as usually, by
\[
(V^*)^i := (V^{-i})^*, \quad d_{V^*} := (d_V)^*,
\]
and for a finite dimensional vector space $W$, by $\text{Det}(W)$ we mean the top exterior power of $W$. If $T$ is a finite tree, $\text{Det}(T)$ is meant to be the top exterior power of $k^{\text{Edge}(T)}$, and by $|T|$ we denote the number of its edges.

Let $P$ be a differential graded operad such that each component $P(n)$ is finite dimensional as a $k$-vector space and $P(1)$ is a semisimple $k$-algebra. Such an operad will be called admissible. For $n \geq 2$, we construct a complex
\[
P(n)^* \otimes \text{Det}(k^n) \xrightarrow{\delta_1} \bigoplus_{|T| = 1} P(T)^* \otimes \text{Det}(\text{Edge}(T)) \xrightarrow{\delta_2} \ldots \xrightarrow{\delta_n} \bigoplus_{|T| = n-2} P(T)^* \otimes \text{Det}(\text{Edge}(T)),
\]
where the direct sums are taken over isomorphism classes of rooted oriented reduced trees with $n$ leaves. The differential $\delta$ is determined by its matrix elements
\[
\delta_{T,T'}: P(T)^* \otimes \text{Det}(\text{Edge}(T')) \to P(T)^* \otimes \text{Det}(\text{Edge}(T)),
\]
where $T,T'$ are trees with $n$ leaves and $|T'| = i$, $|T| = i + 1$. Namely, we take
\[
\delta_{T,T'} = \begin{cases} 
(\gamma_{T/e})^* \otimes l_e, & T' = T/e \\
0, & T' \neq T/e \end{cases}
\]
for any edge $e$,
\[
\text{where } \gamma_{T/e}: P(T) \to P(T') \text{ is naturally induced by the operad structure on } P \text{ and } l_e: \text{Det}(T') \to \text{Det}(T) \text{ is defined by the formula}
\]
\[
l_e(f_1 \wedge \cdots \wedge f_{i-1}) = e \wedge f_1 \wedge \cdots \wedge f_{i-1}.
\]
One can verify that $\delta^2 = 0$ and moreover, $\delta$ commutes with the differential $d^*$ on $P(T)^*$. Thus for each $n \geq 2$, the complex defined above is a bicomplex. We denote its total complex with the differential $d^* + \delta$ by $C'(P)(n)$ and set $C'(P)(1) := P(1)^{op}$. Each $C'(P)(n)$ has an action of $\Sigma_n$ inherited from its action on $P(n)$, and the following statement holds:

**Theorem 3.8.** ([GK94] Theorem 3.2.7) *The $\Sigma$-module $C'(P) := \{C'(P)(n)\}_{n \geq 1}$ has a natural structure of a differential graded operad.*

**Definition 3.9.** The differential graded dual (or simply, dg-dual) of $P$ is the operad
\[
D(P) := C(P) \otimes \Lambda = \{C(P)(n) \otimes \Lambda(n)\},
\]
where $\Lambda(n)$ is a one-dimensional $k$-vector space placed in degree $1-n$ and equipped with the sign action of $\Sigma_n$. 
For any operad $Q$, the dg-dual operad $\mathbf{D}(Q)$ is almost free (that is, free as a linear operad after forgetting the differential structure). The following theorem states that the double dg-dual provides a resolution of $\mathcal{P}$:

**Theorem 3.10.** ([GK94] Theorem 3.2.16) There is a canonical quasi-isomorphism $\mathbf{D}(\mathbf{D}(\mathcal{P})) \to \mathcal{P}$.

**Remark 3.11.** A slightly modified version of the above construction is presented in [GJ94],[LV09], where a pair of adjoint functors

$$B : \{\text{dg-operads}\} \leftarrow \{\text{dg-cooperads}\} : \Omega$$

is constructed, and the resolution of $\mathcal{P}$ appears in the form $\Omega(B(\mathcal{P})) \sim \mathcal{P}$.

**Example 3.12.** As we already mentioned in 1.29, a Koszul operad $\mathcal{P}$ is characterized by the property $\mathbf{D}(\mathcal{P}) \simeq \mathcal{P}^!$. Hence, due to the duality $\mathbb{C}om = \mathbb{L}ie^!$, a cobar resolution of the Lie operad is $\mathbb{L}ie_{\infty} := \mathbf{D}(\mathbb{L}ie^!) = \mathbf{D}(\mathbb{C}om)$. Since $\mathbb{C}om$ has a very simple structure (recall that $\mathbb{C}om(n) = k$ for all $n \geq 1$), it is rather easy to give an explicit description of $\mathbb{L}ie_{\infty}$, which turns out to be a certain tree complex (cf. [GK94, 3.2.15]).

3.13. **Structure of algebras over the differential graded dual.** For an admissible dg-operad $\mathcal{P}$, the following theorem, which is implicit in [GK94],[GJ94], gives a description of $\mathbf{D}(\mathcal{P})$-algebras.

**Theorem 3.14.** Let $V$ be a finite dimensional differential graded $k$-vector space and $\mathcal{P}$ be an operad in $\text{dgVect}_k$. Then there is a differential graded Lie algebra structure on $\bigoplus_{n \geq 1} (\mathcal{P}(n) \otimes V^\otimes n+1)^{\Sigma_n}$ and solutions of the Maurer-Cartan equation $ds + \frac{1}{2}[s,s] = 0$ set up in this algebra are in one-to-one correspondence with the structures of $\mathbf{D}(\mathcal{P})$-algebras on $V$.

In particular, for a Koszul operad $\mathcal{P}$, it implies that a structure of the strongly homotopy $\mathcal{P}$-algebra on $V$ is determined by a solution of the Maurer-Cartan equation set up in $\bigoplus_{n \geq 1} (\mathcal{P}^!(n) \otimes V^\otimes n+1)^{\Sigma_n}$.

4. **Feynman transform of a modular operad**

In this section we describe an analogue of the cobar construction for modular operads, which is called the Feynman transform. As in the case of the cobar construction, the definition of the Feynman transform will involve certain changes in grading and signs of the underlying differential graded vector spaces. In order to keep track on that, we will need the notion of twisting for modular operads. The Feynman transform will be defined then as a free modular operad with a certain twist.

4.1. **Twisted modular operads.**

**Definition 4.2.** A cocycle $D$ is a family of functors $D_{g,n} : \Gamma(g,n) \to \text{dgVect}_k$ defined for all $g,n \geq 0$ subject to the following conditions (we omit the subscripts for $D_{g,n}$):

(a) dim(D(\Gamma)) = 1 for all $\Gamma \in \Gamma(g,n)$ and $D(*_{g,n}) = k$.

(b) For any morphism of stable graphs $f : \Gamma \to \Gamma/J$, there is an isomorphism

$$\nu_f : D(\Gamma/J) \bigotimes_{v \in \text{Vert}(\Gamma/J)} D(f^{-1}(v)) \to D(\Gamma)$$
If $f$ is determined by a family of $\Sigma$-modules, a family of morphisms from $\nu_1$ to $E$, called the coboundary $g$. Let $\nu_1$ be defined to be $\text{Det}(\nu_1) \otimes \nu_1$. For cocycles $D$ and $E$, we define $D^n(\nu_1) := D(\nu_1)^{\otimes n}$, $D^{-1}(\nu_1) := D(\nu_1)^*$ and $DE(\nu_1) := D(\nu_1) \otimes E(\nu_1)$.

**Example 4.3.** Let $V$ be a $k$-vector space. We define $\text{Det}(V)$ to be $\Lambda^{\dim V}(V)[\dim V]$ - the top exterior power of $V$ concentrated in degree $(-\dim V)$. It should not be confused with $\text{Det}(V)$ defined in 3.7, where no shift in grading is performed. For a finite set $S$, $\text{Det}(S)$ is defined to be $\text{Det}(k^S)$.

The following two cocycles will be used later:

- $\mathcal{K}(\nu_1) := \text{Det}(\text{Edge}(\nu_1))$
- $\mathcal{L}(\nu_1) := \text{Det}(\text{Flag}(\nu_1)) \otimes \text{Det}(\text{Leg}(\nu_1))$.

**Definition 4.4.** Let $s$ be a stable $\Sigma$-module with the property that $\dim_k s(g, n) = 1$ for all $g$ and $n$. Then $s$ gives rise to a cocycle $D_s$ defined by

$$D_s(\nu_1) := s(g, n) \otimes \bigotimes_{v \in \text{Vert}(\nu_1)} s^{-1}(g(v), |\text{Leg}(v)|), \quad \nu_1 \in \Gamma(g, n)$$

called the coboundary of $s$.

Recall that an operad structure on a $\Sigma$-module $\mathcal{E}$ is determined by a structure map $\mathcal{M}\mathcal{E} \to \mathcal{E}$, where $\mathcal{M}$ is the monad of stable graphs. Such a structure map boils down to a family of morphisms from

$$\mathcal{M}\mathcal{E}(g, n) \simeq \bigoplus_{\Gamma \in \{\text{Iso}(g, n)\}} \mathcal{E}(\Gamma)_{\text{Aut}(\Gamma)}$$

to $\mathcal{E}(g, n)$ defined for all $g, n \geq 0$ and satisfying certain compatibility conditions.

Similarly, given a cocycle $D$, a twisted modular $D$-operad (or simply $D$-operad) $\mathcal{M}_D\mathcal{E}$ is determined by a family of $\Sigma$-modules

(7) $$\mathcal{M}_D\mathcal{E}(g, n) \simeq \bigoplus_{\Gamma \in \{\text{Iso}(g, n)\}} (D(\Gamma) \otimes \mathcal{E}(\Gamma))_{\text{Aut}(\Gamma)}$$
and morphisms $\alpha^\Gamma : D(\Gamma) \otimes \mathcal{E}(\Gamma) \to \mathcal{E}(g,n)$ defined for all $g, n \geq 0$, $\Gamma \in \Gamma(g,n)$ subject to basically the same compatibility conditions as for ordinary modular operads, but with twisting taken into account.

4.5. **Feynman transform.** Let $\mathcal{P}$ be a modular operad and $D$ be a cocommutative. The dualizing cocycle $D^\vee$ is the cocommutative $K D^{-1}$, where $K$ is as defined in 4.3. We are going to define a modular differential graded $D^\vee$-operad $\mathcal{F}_D \mathcal{P}$ with a differential $d_{\mathcal{F}}$, which will play the role of a dg-dual for the modular operad $\mathcal{P}$.

Recall that for a differential graded operad $\mathcal{Q}$, the underlying $\Sigma$-module of the dg-dual $D(\mathcal{Q})$ is the free $\Sigma$-module generated by $\mathcal{Q}^\vee[-1]$ [GK94, Lemma 3.2.8], and the differential on each $D(\mathcal{Q})(n)$ splits as $d^* + \delta$ (cf. subsection 3.7), where $d$ comes from the original differential on $\mathcal{Q}$. $\delta$ is defined in terms of sums over trees. Similarly, for a modular $D$-operad $\mathcal{P}$, the $\Sigma$-module underlying $\mathcal{F}_D \mathcal{P}$ is taken to be $\mathcal{M}_{D^\vee} \mathcal{P}^*$ with the differential $d_{\mathcal{F}} := \partial^* + \delta$, where $\partial^*$ and $\delta$ are defined as follows.

According to (7), it suffices to determine the actions of $\partial^*$ and $\delta$ on $D^\vee(\mathcal{Q}) \otimes \mathcal{P}^*(\Gamma)$ for $\Gamma \in \Gamma(g,n)$, $g, n \geq 0$. For $\partial^*$, we take $\partial^*(x \otimes y) := x \otimes d^*(y)$, where $d^*$ is the differential on $\mathcal{P}^*$.

Now, consider $\Gamma' \to \Gamma'/e$. The morphism $\Gamma' \to \Gamma'/e$ gives rise (via the modular $D$-operad structure on $\mathcal{P}$) to a map

$$D(\Gamma') \otimes \mathcal{P}(\Gamma') \to D(\Gamma'/e) \otimes \mathcal{P}(\Gamma'/e).$$

Dualizing gives

$$D^{-1}(\Gamma'/e) \otimes \mathcal{P}^*(\Gamma'/e) \to D^{-1}(\Gamma') \otimes \mathcal{P}^*(\Gamma')$$

and, moreover, this descends to

$$(\gamma_{\Gamma' \to \Gamma'/e})^*: (D^{-1}(\Gamma'/e) \otimes \mathcal{P}^*(\Gamma'/e))_{\text{Aut}(\Gamma'/e)} \to (D^{-1}(\Gamma') \otimes \mathcal{P}^*(\Gamma'))_{\text{Aut}(\Gamma')},$$

The same morphism $\Gamma' \to \Gamma'/e$ induces also a map

$$\eta^\gamma_{\Gamma'} : \mathcal{K}(\Gamma'/e) = \text{Det}(\text{Edge}(\Gamma') \setminus \{e\}) \to \text{Det}(\text{Edge}(\Gamma')) = \mathcal{K}(\Gamma'),$$

and we define

$$\delta\bigg|_{D^\vee(\Gamma) \otimes \mathcal{P}^*(\Gamma)} := \sum_{\Gamma' \in \Gamma(g,n)} \gamma_{\Gamma' \to \Gamma'/e}^* \otimes (\gamma_{\Gamma' \to \Gamma'})^*. \tag{8}$$

**Theorem 4.6.** ([GK98] Theorem 5.2)

1. $d_{\mathcal{F}}^2 = (\partial^* + \delta)^2 = 0$.
2. $\mathcal{F}_D \mathcal{P}$ is a modular $D^\vee$-twisted differential graded operad.
3. $\mathcal{F}_D$ is a homotopy functor: a quasi-isomorphism of modular $D$-operads $f : \mathcal{P} \to \mathcal{Q}$ induces a quasi-isomorphism $\mathcal{F}_D f : \mathcal{F}_D \mathcal{P} \to \mathcal{F}_D \mathcal{Q}$.

The twisted $D^\vee$-operad $\mathcal{F}_D \mathcal{P}$ is called the **Feynman transform** of $\mathcal{P}$. The term is meant to emphasize a formal similarity between expression (8), where the sum is taken over representatives of isomorphism classes of graphs, and the formulas expressing certain integrals with respect to the Gaussian measure in terms of sums over graphs or, in physicists terminology, “Feynman diagrams” (cf. appendix).

**Example 4.7.** ([Kap98] Example 4.5) Recall that the associative operad $\text{Ass}$ has a cyclic structure. Let $\mathcal{P}$ be the modular operad $\text{Ass}$ concentrated in genus zero (as in example 2.23). One can show that

$$\mathcal{F}_D(\mathcal{P})(g,n) = \bigoplus_{2g-2+n=\chi} C_n(|M_{g,n}|/\Sigma_n, \mathbb{C}),$$
where $|\mathcal{M}_{g,n}|$ is the coarse moduli space of smooth genus $g$ curves with $n$ punctured points and $C_*$ is the chain complex with respect to Penner’s cell decomposition labelled by ribbon graphs (or “fat graphs” [Pen88]).

The following statement is analogous to theorem 3.10.

**Theorem 4.8.** ([GK98] Theorem 5.4) For a modular $D$-operad $P$, the canonical map $\mathcal{F}_{D^\vee}(\mathcal{F}_D P) \to P$ is a quasi-isomorphism.

5. Structure of algebras over the Feynman transform

In this section we give an account of Barannikov’s theorem that describes the structure of algebras over the Feynman transform in terms of solutions of the quantum master equation. By definition, such an equation is supposed to be set up in some Batalin-Vilkovisky algebra (or $BV$-algebra, for short). So we begin by describing this algebraic structure.

5.1. BV-algebras. A differential graded BV-algebra is a differential graded Gerstenhaber algebra (cf. subsection 1.40) equipped with an extra square-zero linear map, whose failure of being a derivation is measured by the Lie bracket. In details,

**Definition 5.2.** A differential graded BV-algebra is a differential graded vector space $(V,d)$, with $d$ of degree $(-1)$, equipped with

1. a commutative associative bilinear product $\cdot : V \otimes V \to V$ of degree 0;
2. a graded-commutative Lie bracket $\{,\} : V \otimes V \to V$ of degree $(-1)$;
3. a linear map $\Delta : V \to V$ of degree $(-1)$

subject to the following conditions:

(a) $\Delta^2 = 0$ and $\Delta d + d \Delta = 0$;
(b) the product $\cdot$ and the bracket $\{,\}$ are related by the graded Leibniz rule;
(c) for $a,b \in V$, $\{a,b\} = \Delta(a \cdot b) - (\Delta(a) \cdot b + (-1)^{|a|} a \cdot \Delta(b))$;
(d) for $a,b \in V$, $\Delta(\{a,b\}) = \{\Delta(a), b\} + (-1)^{|a|-1}\{a, \Delta(b)\}$.

**Example 5.3.** (1) For a differential graded Lie algebra $\mathfrak{g}$, the exterior algebra $\wedge^\bullet(\mathfrak{g})$ is a differential graded BV-algebra with $\Delta$ being the standard Chevalley-Eilenberg differential.
(2) The Gerstenhaber algebra structure on multivector fields (cf. example 1.42) extends to a BV-algebra.
(3) Hochschild cohomology of a unital, associative algebra with a symmetric, invariant non-degenerate inner product is a BV-algebra.[Tra08]
(4) It is a result due to M.Chas and D.Sullivan[CS99] that the homology of the space of free loops $LX$ on a smooth closed oriented manifold $X$ is a BV-algebra.

**Remark 5.4.** While Gerstenhaber algebras are governed by the homology operad of little 2-cubes, or equivalently, little 2-disks (cf. subsection 1.40), Batalin-Vilkovisky algebras (without differential grading) are algebras over the operad of framed little 2-disks [Get94]. Resolutions of the BV-operad and homotopy BV-algebras are discussed in [GCTV09].

**Definition 5.5.** A quantum master equation in a differential graded BV-algebra is the equation

$$dm + \Delta m + \frac{1}{2}\{m,m\} = 0.$$

**Remark 5.6.** Historically, first explicit examples of BV-algebras appeared in the work of I.Batalin and G.Vilkovisky on gauge fixing in quantum field theory. Without giving any details (for it would be too much of a digression into that exciting field), we would just
mention that in this basic example, the BV-structure was set up on the space of regular functions \( \mathcal{F}(W \oplus \prod W^*) \), where \( W \) is a superspace and \( \prod \) is the parity changing functor. The bilinear product is the pointwise product on \( \mathcal{F}(W \oplus \prod W^*) \), the bracket \( \{ \Phi, \Psi \} \) is a sum of properly graded expressions of the form \( \frac{\partial \Phi}{\partial x_i} \frac{\partial \Psi}{\partial x_i} - (-1)^{(|\Phi|+1)(|\Psi|+1)} \frac{\partial \Phi}{\partial x^i} \frac{\partial \Psi}{\partial x_i} \) and the operator \( \Delta \) is the Laplacian \( \sum_i \frac{\partial^2 \Phi}{\partial x^i \partial x_i} \). For more details on this example and general information on applications of BV-structures in physics an interested reader is referred to [Sch93], [Wit90], [Zwi93], [Get94], [ASZK97].

Since then, other instances of BV-algebras have been discovered in various other fields. A list of references can be found in [DCV11].

5.7. Twisted modular endomorphism operad. Let \((V,d)\) be a differential graded \(k\)-vector space, \(l \in \mathbb{Z}\) and \(B : V \otimes V \to k[-l] \) be a graded symmetric bilinear map such that \(B(u,v) = 0\) unless \(|u| + |v| = l\). We are going to define the modular endomorphism \(K^l\)-operad \(\mathcal{E}nd_V\) by giving a twist to the construction of example 2.24.

As before, for \(g,n \geq 0\), we set \(\mathcal{E}nd_V(g,n) := V^{\otimes n}\). Now, in order to give the \(\Sigma\)-module \(\{\mathcal{E}nd_V(g,n)\}_{g,n \geq 0}\) a structure of a twisted modular operad, we define for each \(\Gamma \in \Gamma(g,n)\) a structure morphism \(\alpha_{\Gamma} : \mathcal{K}^l(\Gamma) \otimes \mathcal{E}nd_V(\Gamma) \to \mathcal{E}nd_V(g,n)\) as follows. Let \(B^{\text{Edge}(\Gamma)}\) be as in 2.24. Then

\[
\alpha_{\Gamma} : \mathcal{K}^l(\Gamma) \otimes \mathcal{E}nd_V(\Gamma) = \text{Det}(\text{Edge}(\Gamma))^{|\Gamma|} \otimes V^{\text{Flag}(\Gamma)} \xrightarrow{id \otimes B^{\text{Edge}(\Gamma)}} \text{Det}(\text{Edge}(\Gamma))^{|\Gamma|} \otimes V^{\text{Leg}(\Gamma)}[-l \cdot |\text{Edge}(\Gamma)|] \\
\simeq \text{Det}(\text{Edge}(\Gamma))^{|\Gamma|} \otimes V^{\otimes n}[-l \cdot |\text{Edge}(\Gamma)|] \simeq V^{\otimes n} \simeq \mathcal{E}nd_V(g,n).
\]

Similarly, for a graded antisymmetric bilinear map \(B : V \otimes V \to k[-l]\), one constructs a twisted endomorphism \(\mathcal{K}^{l-2} \otimes \mathcal{L}\)-operad \(\mathcal{E}nd_V\).

5.8. Algebras over \(\mathcal{F}_D(\mathcal{P})\). Let \(V\) be finite dimensional differential graded \(k\)-vector space with a form \(B\) as in the previous subsection, \(D\) be the cocycle \(\mathcal{K}^{1-l}\) and \(\mathcal{P}\) be a \(D\)-operad with \(\text{dim} \mathcal{P}(g,n) < \infty\) for all \(g,n \geq 0\). Notice that \(D' = \mathcal{K}^l\). Thus \(\mathcal{E}nd_V\) is a \(\mathcal{F}_D\)-operad as well.

By definition, a \(\mathcal{F}_D\mathcal{P}\)-algebra structure on \(V\) is determined by a morphism of differential graded modular \(\mathcal{D}'\)-operads \(\tilde{m} : \mathcal{F}_D\mathcal{P} \to \mathcal{E}nd_V\). That means that for each \(\Gamma \in \Gamma(g,n)\) there is a morphism \(\tilde{m} : \mathcal{F}_D\mathcal{P}(\Gamma) \to \mathcal{E}nd_V(\Gamma) \simeq V^{\text{Flag}(\Gamma)}\) and the diagrams

\[
\begin{array}{c}
\mathcal{F}_D\mathcal{P}(\Gamma) \xrightarrow{\tilde{m}} \mathcal{E}nd_V(\Gamma) \\
d_x | \downarrow \quad \downarrow d_{\mathcal{E}nd_V} \\
\mathcal{F}_D\mathcal{P}(\Gamma) \xrightarrow{\tilde{m}} \mathcal{E}nd_V(\Gamma)
\end{array}
\]

\[
\begin{array}{c}
\mathcal{F}_D\mathcal{P}(\Gamma') \xrightarrow{\tilde{m}} \mathcal{E}nd_V(\Gamma') \\
\gamma_{\mathcal{D}' \to \mathcal{D}} \uparrow \quad \downarrow \gamma_{\mathcal{E}nd_V \to \mathcal{D}} \\
\mathcal{F}_D\mathcal{P}(\Gamma) \xrightarrow{\tilde{m}} \mathcal{E}nd_V(\Gamma)
\end{array}
\]
of $\gamma^\ast (\bigotimes_{v \in \text{Vert}(\Gamma')} \hat{m}_v, \text{Leg}(v))$ to be a morphism of $D^V$-operads. This equation will constitute then a necessary and sufficient condition for $\hat{m}$ to be restated in the form of the quantum master equation on a certain differential graded $\ast'$-operad $H/e\partial$ and $\ast'$.

If $\Gamma = \ast_{g,n}$ holds. It suffices to check this equation on the generators of $\mathcal{F}_D\mathcal{P}$, that is, on the spaces $\mathcal{P}^\ast(\ast_{g,n})$ for $g,n \geq 0$.

In what follows, we spell out equation (12) on $\mathcal{P}^\ast(\ast_{g,n}) \simeq \mathcal{P}^\ast(g,n)$ and show how it can be restated in the form of the quantum master equation on a certain differential graded BV-algebra. This equation will constitute then a necessary and sufficient condition for $\hat{m}$ to be a morphism of $D^V$-operads.

**Step 1.** On the right-hand side of (12) the differential $d_{\mathcal{F}}$ is, by definition,

$$d_{\mathcal{F}}|_{(D^V(\Gamma) \otimes \mathcal{P}^\ast(\Gamma))_{\text{Aut}(\Gamma)}} = \partial^\ast + \sum_{H \in \{\Gamma(g,n)\}} \gamma^H_{\ast_{g,n}} \otimes (\gamma^H_{\mathcal{F}(\Gamma)})^\ast,$$

If $\Gamma = \ast_{g,n}$, then

$$(D^V(\Gamma) \otimes \mathcal{P}^\ast(\Gamma))_{\text{Aut}(\Gamma)} = k \otimes \mathcal{P}^\ast(\ast_{g,n})$$

and $\partial^\ast$ in (12) is simply the original differential $d^\ast$ on $\mathcal{P}^\ast$.

To handle the sum over graphs in (13) for $\Gamma = \ast_{g,n}$, observe that a stable graph $H$ such that $H/e \simeq \ast_{g,n}$ must be of one of the following two types:

(a) a unique (up to isomorphism) graph with one vertex of genus $g - 1$, one self-intersecting edge (a loop) $e = (f f')$ and $n$ external legs. We denote such a graph by $\Gamma_{g,n}$;

(b) a graph with two vertices of genera $g_1$ and $g_2$, one edge $e = (f f')$ and $|I_1| + |I_2|$ external legs labeled by elements of some finite sets $I_1, I_2$. Such a graph will be denoted by $\Gamma_{g_1,g_2,I_1,I_2}$. 

- For any $\Gamma' \in \Gamma(g,n)$ and any stable graph morphism $\Gamma' \to \Gamma$. In particular, if $\Gamma = \ast_{g,n}$, then $\mathcal{E}\text{nd}_V(\Gamma) \simeq V^\otimes n$, $D^V(\ast_{g,n}) = k$ and the latter diagram takes the form

$$\mathcal{F}_D\mathcal{P}(\Gamma') = \left( D^V(\Gamma') \otimes \bigotimes_{v \in \text{Vert}(\Gamma')} \mathcal{P}^\ast(g(v), \text{Leg}(v)) \right)_{\text{Aut}(\Gamma')} \xrightarrow{id \otimes \bigotimes_{v \in \text{Vert}(\Gamma')} \hat{m}_v} D^V(\Gamma') \otimes V^\otimes \text{Flag}(\Gamma') \xrightarrow{\gamma^\ast_{\Gamma' \to \ast_{g,n}}} \mathcal{P}^\ast(g,n) \xrightarrow{\tilde{m}} V^\otimes n.$$
In both cases, the map $\eta^H_\varepsilon : \mathcal{K}(H/e) \to \mathcal{K}(H)$ simply shifts grading by (-1):

$$k = \mathcal{K}(g_{g,n}) = \mathcal{K}(H/e) \xrightarrow{\eta^H_\varepsilon} \mathcal{K}(H) = \text{Det} (\text{Edge}(H)) = \text{Det}(\{e\}) = k \cdot e[1].$$

Denoting the one-dimensional space $k \cdot e$ placed in degree (-1) by $e[1]$, we rewrite (13) as

$$d_f = d^* + e[1] \otimes (\gamma^p_{\Gamma_{g,n}})^* + \frac{1}{2} \sum_{I_1 \sqcup I_2 = \{1,\ldots,n\}} e[1] \otimes (\gamma^p_{\Gamma_{g_{91,2},I_1,I_2}})^*,$$

The factor $\frac{1}{2}$ in the above sum appears, because isomorphic graphs $\Gamma_{g_{1,2},I_1,I_2}$ and $\Gamma_{g_{2,1},I_1,I_2}$ represent the same class in (13).

The maps $(\gamma^p_{\Gamma_{g,n}})^* : \mathcal{P}^*(g,n) \to \mathcal{P}^*(\Gamma_{g,n})$ and $(\gamma^p_{\Gamma_{g_{91,2},I_1,I_2}})^* : \mathcal{P}^*(g,n) \to \mathcal{P}^*(\Gamma_{g_{91,2},I_1,I_2})$ in (14) are duals of the standard maps $\gamma^p_{\Gamma_{g_{91,2},I_1,I_2}} : \mathcal{P}^*(g,n) \to \mathcal{P}^*(\Gamma_{g_{91,2},I_1,I_2})$ and $\gamma^p_{\Gamma_{g_{91,2},I_1,I_2}} : \mathcal{P}^*(g_{91,2},I_1,I_2) \to \mathcal{P}^*(g_{91,2},I_1,I_2)$ coming from the twisted modular operad structure of $\mathcal{P}$. Notice (cf. formula (4) of subsection 2.15) that we have

$$\mathcal{P}^*(\Gamma_{g,n}) = \mathcal{P}^*(g - 1, \{1,\ldots,n\}) \sqcup \{f,f'\}$$
$$\mathcal{P}^*(\Gamma_{g_{91,2},I_1,I_2}) = \mathcal{P}^*(g_1, I_1 \{f\}) \otimes \mathcal{P}^*(g_2, I_2 \{f'\}).$$

**Step 2.** First, for a set $I$ of cardinality $n$, we extend, in the usual way, each $\hat{m}_{g,I}$ to a map $\hat{m}_{g,I} : \mathcal{P}^*(g, I) \to V^{\otimes I}$.

Now, composing (14) with $\hat{m}$ defined by (10), we obtain

$$\hat{m} \circ d_f = \hat{m} \circ d^* + \hat{m} \circ (e[1] \otimes (\gamma^p_{\Gamma_{g,n}})^*) + \frac{1}{2} \sum_{I_1 \sqcup I_2 = \{1,\ldots,n\}} \hat{m} \circ (e[1] \otimes (\gamma^p_{\Gamma_{g_{91,2},I_1,I_2}})^*)$$

$$= \hat{m}_{g,n} \circ d^* + (\gamma^p_{\Gamma_{g,n}}) \circ \hat{m}_{g-1,\{1,\ldots,n\} \sqcup \{f,f'\}} \circ (e[1] \otimes (\gamma^p_{\Gamma_{g,n}})^*)
$$

$$+ \frac{1}{2} \sum_{I_1 \sqcup I_2 = \{1,\ldots,n\}} (\gamma^p_{\Gamma_{g_{91,2},I_1,I_2}} \circ \hat{m}_{g_1, I_1 \{f\} \otimes \hat{m}_{g_2, I_2 \{f'\}}}) \circ (e[1] \otimes (\gamma^p_{\Gamma_{g_{91,2},I_1,I_2}})^*).$$

Our main equation (12) becomes

$$d_{V \otimes n} \circ \hat{m}_{g,n} = \hat{m}_{g,n} \circ d^* + (\gamma^p_{\Gamma_{g,n}}) \circ (e[1] \otimes (\hat{m}_{g-1,\{1,\ldots,n\} \sqcup \{f,f'\}}) \circ (\gamma^p_{\Gamma_{g,n}})^*)
$$

$$+ \frac{1}{2} \sum_{I_1 \sqcup I_2 = \{1,\ldots,n\}} (\gamma^p_{\Gamma_{g_{91,2},I_1,I_2}} \circ (e[1] \otimes (\hat{m}_{g_1, I_1 \{f\} \otimes \hat{m}_{g_2, I_2 \{f'\}}}) \circ (\gamma^p_{\Gamma_{g_{91,2},I_1,I_2}})^*).$$

**Step 3.** Since $V$ and $\mathcal{P}^*(g,n)$ are finite-dimensional, then a family of $\Sigma_n$-equivariant morphisms $\{m_{g,n} : \mathcal{P}^*(g,n) \to V^{\otimes n}\}_{g,n \geq 0}$ gives rise, after taking duals, to a family of degree zero elements $\{m_{g,n} \in (V^{\otimes n} \otimes \mathcal{P}(g,n))^\Sigma_n\}_{g,n \geq 0}$. We would like to dualize equation (15) as well and rewrite it in terms of $m_{g,n}$. 

Recall first that $\mathcal{E}nd_{\mathcal{V}}$ is a $\mathcal{K}^{l}$-operad. Since $\Gamma_{g,n}, \Gamma_{g_{1},g_{2},I_{1},I_{2}}$ both have only one internal edge, then $\mathcal{K}^{l}(\Gamma_{g,n}) = \mathcal{K}^{l}(\Gamma_{g,n}) = k[l]$. Then $\gamma_{\mathcal{E}nd_{\mathcal{V}}} : \mathcal{E}nd_{\mathcal{V}}(\Gamma_{g,n}) \to V^\otimes n$ is the contraction $(V(f) \otimes V^\otimes n)[l] \to V^\otimes n$ done by applying the form $B$ to factors corresponding to the flags $f$ and $f'$. Similarly, $\gamma_{\mathcal{E}nd_{\mathcal{V}}} : \mathcal{E}nd_{\mathcal{V}}(\Gamma_{g_{1},g_{2},I_{1},I_{2}}) \to V^\otimes I$. Both of these maps will be denoted by $B_{f,f'}$.

We define

$$\phi_{f,f'}^p : \mathcal{P}(g_{1}, I_{1} \Pi \{f\}) \otimes \mathcal{P}(g_{2}, I_{2} \Pi \{f\}) \to \mathcal{P}(g_{1} + g_{2}, |I_{1}| + |I_{2}|)[l - 1], \quad \phi_{f,f'}^p = \gamma_{\mathcal{E}nd_{\mathcal{V}}}^p(I_{g_{1},g_{2},I_{1},I_{2}})(e[1 - l])$$

(16)

$$\xi_{f,f'}^p : \mathcal{P}(g - 1, \{1, \ldots, n\} \Pi \{f, f'\}) \to \mathcal{P}(g, n)[l - 1], \quad \xi_{f,f'}^p = \gamma_{\mathcal{E}nd_{\mathcal{V}}}^p(I_{g, n})(e[1 - l])$$

and obtain from (15) the equation

$$(d_{\mathcal{P}} + d_{V})m_{g,n} - (B_{f,f'} \otimes \xi_{f,f'}^p)(m_{g-1, \{1, \ldots, n\} \Pi \{f, f'\}})$$

$$- \frac{1}{2} \sum_{\{1, \ldots, n\} = I_{1} \Pi I_{2}} (B_{f,f'} \otimes \phi_{f,f'}^p)(m_{I_{1} \Pi \{f\}} \otimes m_{I_{2} \Pi \{f'\}}),$$

where $d_{\mathcal{P}}, d_{V}$ are differentials on $\mathcal{P}$ and $V$ respectively.

After introducing a formal variable $h$ and taking $m_{I} = \sum_{g \geq 0} h^{g}m_{g,I}$, we can rewrite the last equation as

$$(d_{\mathcal{P}} + d_{V})m_{n} - h(B_{f,f'} \otimes \xi_{f,f'}^p)(m_{\{1, \ldots, n\} \Pi \{f, f'\}})$$

$$- \frac{1}{2} \sum_{\{1, \ldots, n\} = I_{1} \Pi I_{2}} (B_{f,f'} \otimes \phi_{f,f'}^p)(m_{I_{1} \Pi \{f\}} \otimes m_{I_{2} \Pi \{f'\}}).$$

(17)

By taking a bijection $\{1, \ldots, n\} \Pi \{f, f'\} \leftrightarrow \{1, \ldots, n, n + 1, n + 2\}$, we identify

$$(B_{f,f'} \otimes \xi_{f,f'}^p)(m_{\{1, \ldots, n\} \Pi \{f, f'\}})$$

in (17) with $(B_{n+1,n+2} \otimes \xi_{n+1,n+2}^p)m_{n}$, where $m_{n} = \sum_{g \geq 0} m_{g,n}$.

Similarly, after identifications

$I_{1} \Pi \{f\} = \{1, \ldots, n_{1}, f\} \leftrightarrow \{1, \ldots, n_{1}, n_{1} + 1\}$

and

$I_{2} \Pi \{f'\} = \{n_{1} + 1, \ldots, n_{1} + n_{2}, f'\} \leftrightarrow \{1, \ldots, n_{2}, n_{2} + 1\}$

$\phi_{f,f'}^p$ becomes a map $\mathcal{P}(g_{1}, n_{1} + 1) \otimes \mathcal{P}(g_{2}, n_{2}) \to \mathcal{P}(g_{1} + g_{2}, n_{1} + n_{2})[l - 1]$, which we denote by $o_{\mathcal{P}}$. Equation (17) becomes

$$(d_{\mathcal{P}} + d_{V})m_{n} - h(B_{n+1,n+2} \otimes \xi_{n+1,n+2}^p)(m_{n+1})$$

$$- \frac{1}{2} \sum_{n_{1} + n_{2} = n} \sum_{\sigma \in \Sigma_{n}} \sigma(B_{n+1,n+2} \otimes o_{\mathcal{P}})(m_{n_{1}+1} \otimes m_{n_{2}+1})$$

(18)

**Step 4.** One verifies that the formulas

$$\{s_{1}, s_{2} \} := \frac{(-1)^{|s_{1}|\cdot |s_{2}|}}{n_{1}! \cdot n_{2}!} \sum_{\sigma \in \Sigma_{n}} \sigma(o_{\mathcal{E}nd_{\mathcal{V}}} \otimes o_{\mathcal{P}})(s_{1} \otimes s_{2}), \quad s_{i} \in (V^\otimes n_{i} \otimes \mathcal{P}(g_{i}, n_{i}+1))_{\Sigma_{n_{i}+1}}$$

and

$$\Delta s := (-1)^{|s|} \xi_{n-1,n} \otimes \xi_{n-1,n}^p s, \quad s \in (V^\otimes n \otimes \mathcal{P}(g, n))_{\Sigma_{n}}$$
define a graded Lie bracket and a differential (both of degree \((-1)\)) on

\[ F = \bigoplus_{g,n \geq 0} (V \otimes \mathbb{P}(g,n))^\Sigma_n. \]

The differential \(d\) on \(F\) is defined by

\[ dm := (-1)^{|m|+1}(d\mathbb{P} - dV)m. \]

Equation (18) can be written as

\[ dm_n + \hbar \Delta m_{n+2} + \frac{1}{2} \sum_{n_1 + n_2 = n} \{m_{n_1}, m_{n_2}\} = 0 \]

By introducing one more formal variable \(\lambda\) ("the string coupling constant") and setting up the ansatz \(m = \sum_{2g-2+n \geq 0} m_{g,n} h^g \lambda^n\), we finally arrive to the quantum master equation

\[ (19) \]

\[ dm + \hbar \Delta m + \frac{1}{2} \{m, m\} = 0. \]

Thus, we proved

**Theorem 5.9.** ([Bar07] Theorem 1) If \(\mathbb{P}\) is a modular \(K^{1-l}\)-operad, then the \(\mathcal{F}_D\mathbb{P}\)-algebra structures on a differential graded vector space \(V\) with a graded symmetric inner product \(B : V \otimes V \to k[-l]\) of degree \(l\) are in one-to-one correspondence with solutions of the quantum master equation (19), where each \(m_{g,n}\) is a degree zero element of \(\bigoplus_{g,n \geq 0} (V \otimes \mathbb{P}(g,n))^\Sigma_n\).

**Remark 5.10.** An analogous statement holds for a dg-space \(V\) with a graded antisymmetric form \(B\). The difference is that \(\mathbb{P}\) in that case is required to be twisted by \(K^{1-l}\).L.

**Remark 5.11.** The result can be stated in the language of formal non-commutative symplectic geometry as we did it in the introduction. For more details we refer the reader to the original paper [Bar07] and to the work [Gin01].

6. **Further research**

6.1. **Structure of algebras over the dg-dual and over the Feynman transform of a cyclic operad.** For any cyclic operad (say, for a classical operad \(Ass, Com\) or \(Lie\)) there is, on one hand, the underlying ordinary operad and, on the other hand, the corresponding modular operad concentrated in genus zero (cf. example 2.23). Thus, for \(\mathbb{P}\) we may find both the dg-dual operad \(\mathcal{D}\mathbb{P}\) and the Feynman transform \(\mathcal{F}_D\mathbb{P}\). We would like to compare the structures of algebras over \(\mathcal{D}\mathbb{P}\) and \(\mathcal{F}_D\mathbb{P}\). Theorem 3.14 and Barannikov’s theorem 5.9, which describe structures of algebras over \(\mathcal{D}\mathbb{P}\) and \(\mathcal{F}_D\mathbb{P}\), are analogous. So, on the one hand, one may expect that for cyclic operads, solutions to the QME (19) reduce to solutions of the Maurer-Cartan equation of theorem 3.14 by taking the genus zero part (i.e. setting \(\hbar\) to zero in the formal series for \(m\)). On the other hand, that analogy is not strict. There is a subtle point that each direct summand of the DGLA \(\bigoplus_{n \geq 1} (\mathbb{P}(n) \otimes V^\otimes(n+1))^\Sigma_n\) in theorem 3.14 has an extra factor of \(V\) due to the lack of symmetry of ordinary operads in comparison with cyclic ones. The structures coming from solutions of the QME by reduction modulo \(\hbar\) may be called cyclic \(\mathcal{D}(\mathbb{P})\)-algebras. Classical versions of these structures are known as cyclic \(A_\infty\) and cyclic \(L_\infty\)-algebras. An explicit relationship between those and the usual \(A_\infty\) and \(L_\infty\)-algebras is studied in [Ham09],[HL09]. We would like to explore this in the case of general cyclic operads.
6.2. **Model structures on categories of modular operads.** As we mentioned in section 3, model structures on categories of operads are well-understood, thanks to works of V. Hinich [Hin97], C. Berger, I. Moerdijk [BM03], [BM06] and others. Model structures on properads were introduced in [MV09]. But to our knowledge, no model structures on modular operads have been studied systematically yet, though it looks as an interesting topic to investigate. The desired property for model structure on modular operads is that the double Feynman transform provides (in view of theorem 4.7) a cofibrant resolution.

6.3. **Koszulness for modular operads and minimal models.** The problem here is to make sense of Koszul property for modular operads and see if the Feynman transform of the quadratic dual will be a minimal model.

6.4. **Quantum deformation theory.** There is a general idea that in characteristic zero deformations of objects over commutative parameter spaces are controlled by differential graded Lie algebras or, more generally, by $L_{\infty}$-algebras. By that we mean that a deformation functor $D : Art_k \to \text{Sets}$ (in the sense of Schlessinger) should be isomorphic to the functor $Def_L$ defined by

$$Def_L(R) := \{\text{solutions of the Maurer-Cartan equation for } L \otimes R\}/\{\text{gauge equivalence}\}$$

for some $L_{\infty}$-algebra $L$. This point of view is justified by numerous examples. For instance, deformations of a complex manifold $X$ are controlled by a Dolbeault resolution of the sheaf of multivector elds, which obtains a $L_{\infty}$-structure from the $\bar{\partial}$-operator and the Schouten-Nijenhuis bracket; deformations of an associative algebra $A$ are controlled by the Hochschild cochain complex of $A$, whose $L_{\infty}$-structure is determined by the Hochschild coboundary operator and the Gerstenhaber bracket.

The idea of the quantum deformation theory [Ter10] is to hand over the role of the basic algebraic structure from DGLAs to differential graded BV-algebras, and to replace the Maurer-Cartan equation by the quantum master equation. Within the context of our paper, it appears to us that we have encountered a particular case of this process. We can summarize our vision of it in the following “Rosetta Stone”:

<table>
<thead>
<tr>
<th>Classical</th>
<th>Quantum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rooted trees</td>
<td>Modular operads</td>
</tr>
<tr>
<td>Operads</td>
<td>dg BV-algebras</td>
</tr>
<tr>
<td>DGLAs</td>
<td>Feynman transform</td>
</tr>
<tr>
<td>dg-dual</td>
<td>Quantum(?) deformations of algebras over modular operads</td>
</tr>
<tr>
<td>Maurer-Cartan equation</td>
<td>Quantum master equation</td>
</tr>
<tr>
<td>Deformations of algebras over operads [KS00]</td>
<td></td>
</tr>
</tbody>
</table>

The problems, which arise naturally here, are

1. find BV-counterparts of some classical DGLAs arising in geometric and algebraic contexts;
2. find interpretation of lower-degree terms of solutions of the corresponding QMEs.

**7. Appendix: Sums over graphs in perturbative field theory**

References for the material presented here are [BIZ80] and [Wit99].

In (perturbative) quantum field theory one is interested in evaluating integrals of the form $Z = \int_V e^{-S(v)} dv$ (partition function) and, more generally,

$$\langle f_1, \ldots, f_n \rangle = \frac{1}{Z} \int_V f_1(v) \ldots f_n(v) e^{-S(v)} dv$$
known as the correlation function. Here, $V$ is a $d$-dimensional real vector space, $f_i \in V^*$ and $S(v) = \frac{B(v,v)}{2} - \sum_{m \geq 3} g_m \frac{S_m(v^{\otimes m})}{m!}$, where $B$ is a positive definite symmetric bilinear form and each $S_m$ is a symmetric tensor on $V^*$. Even if such an integral diverges, we would like to find its value as a formal power series in variables $g_i$. It turns out that these power series can be conveniently presented as sums over graphs.

In the simplest case, when $S(v) = \frac{B(v,v)}{2}$, by elementary means one proves

**Lemma 7.1.** (G.-C.Wick) If $n$ is even, then

$$\int_V f_1(v) \cdots f_n(v) e^{-\frac{B(v,v)}{2}} dv = \frac{(2\pi)^{d/2}}{\sqrt{\det(B)}} \sum_{\lambda} B^{-1}(f_{i_1}, f_{i_2}) \cdots B^{-1}(f_{i_{n-1}}, f_{i_n}),$$

where $B^{-1}$ is the inverse form on $V^*$ and the summation is done over all unordered pairings $(i_1, i_2), \ldots (i_{n-1}, i_n)$ of indices from $\{1, \ldots, n\}$. For odd $n$ the above integral vanishes, because the integrand, in this case, is an odd function.

Notice that each pairing can be represented by a graph with $n$ vertices, labeled by numbers from 1 to $n$, and $n/2$ edges connecting vertices with indices forming a pair. We are going to generalize it.

Let $S(v)$ be of the general form as above. Consider a sequence of non-negative integers $\bar{n} = (n_3, \ldots, n_i, \ldots)$, which is eventually zero. Denote by $\Gamma(n, \bar{n})$ the set of all reduced graphs with $n$ vertices of degree one labelled by $1, \ldots, n$ and $n_i$ vertices of degree $i$. For any $\Gamma \in \Gamma(n, \bar{n})$ define a polylinear function $F_\Gamma$ of $f_1, \ldots, f_n$ as follows. To a vertex of degree one labelled by $i$ assign the linear form $f_i$, and to each vertex of degree $m$ attach the tensor $S_m$. Now, consider the tensor product of these $S_m$’s and perform the contractions along edges using the form $B^{-1}$. This is the value of $F_\Gamma(f_1, \ldots, f_n)$. The reader is encouraged to compare this construction with $B^{\text{Edge}}(\Gamma)$ from the definition of the modular endomorphism operad (cf. example 2.24).

**Theorem 7.2.** (R.Feynman)

$$\langle f_1, \ldots, f_n \rangle = \sum_{\bar{n}} \left( \prod_i g_i^{n_i} \right) \sum_{\Gamma \in \Gamma(n, \bar{n})} \frac{1}{|\text{Aut}(\Gamma)|} F_\Gamma(f_1, \ldots, f_n),$$

where $\text{Aut}(\Gamma)$ is the group of automorphism of the graph $\Gamma$ preserving the labellings of the degree one vertices and the second sum is taken over representatives of isomorphism classes of graphs.

**References**


