1. (20 points) (a) Use Euler's method with step size 0.2 to estimate \( y(0.4) \) where \( y(x) \) is the solution of the initial-value problem \( y' = x + y^2, \ y(0) = 0. \)

\[
y_n = y_{n-1} + 0.2 \left( x_{n-1} + y_{n-1}^2 \right); \quad x_0 = 0, \ y_0 = 0.
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>0 + 0.2 (0 + 0^2) = 0</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>0 + 0.2 (0.2 + 0^2) = 0.04</td>
</tr>
</tbody>
</table>

\[ y(0.4) \approx 0.04 \]

(b) Find the solution of the differential equation that satisfies the given initial condition \( \frac{dp}{dt} = \sqrt{pt}, \ p(1) = 2. \)

\[
\frac{dp}{\sqrt{p}} = \int \sqrt{t} \ dt
\]

\[
\int p^{-\frac{1}{2}} dt = \int t^{\frac{1}{2}} dt
\]

\[
2 \ p^{\frac{1}{2}} = \frac{2}{3} t^{\frac{3}{2}} + C
\]

\[
p(1) = 2 \Rightarrow 2\sqrt{2} = \frac{2}{3} (1) + C \Rightarrow 2\sqrt{2} - \frac{2}{3} = C
\]

\[
2\sqrt{p} = \frac{2}{3} t^{\frac{3}{2}} + 2\sqrt{2} - \frac{2}{3}
\]

\[
\sqrt{p} = \frac{1}{3} t^{\frac{3}{2}} + \sqrt{2} - \frac{1}{3} = \Rightarrow p = \left( \frac{1}{3} t^{\frac{3}{2}} + \sqrt{2} - \frac{1}{3} \right)^2
\]
2. (10 points) A tank contains 1000 L of brine with 15kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. Let \( y(t) \) be the amount of salt in the tank at time \( t \). Set up, **BUT DO NOT SOLVE**, a differential equation representing the amount of salt in the tank as a function of time. Include the appropriate initial condition.

Let \( y(t) \) be the amount of salt in the tank at time \( t \):

\[
\frac{dy}{dt} = (\text{rate-in}) - (\text{rate-out})
\]

\[
= (0)(10) - \left( \frac{y(t)}{1000} \right)(10).
\]

Hence

\[
\frac{dy}{dt} = -\frac{y(t)}{100} ; \quad y(0) = 15
\]
3. (30 points) (a) Find the area enclosed by the curve \( x = t^2 - 2t, \quad y = \sqrt{t} \) and the y-axis.

- The curve crosses the y-axis when \( x = 0 \).
- \( 0 = x = t^2 - 2t = t(t-2) \) \( \Rightarrow \) \( t = 0 \) or \( 2 \). 
- \( x_L = t^2 - 2t \) 
- \( y = \sqrt{t} \) \( \Rightarrow \) \( dy = \frac{1}{2\sqrt{t}} \) \( dt \) 

Area = \( \int_{a}^{b} x \, dy = \int_{a}^{b} (x_L - x_L) \, dy \)

\[
= \int_{0}^{2} \left( 0 - (t^2 - 2t) \right) \left( \frac{1}{2\sqrt{t}} \right) \, dt \\
= \int_{0}^{2} \left( \frac{2t - t^2}{2\sqrt{t}} \right) \, dt = \int_{0}^{2} \left( \frac{1}{2} - \frac{t}{2\sqrt{t}} \right) \, dt \\
= \left[ \frac{2}{3} \left( \frac{3}{2} \right) - \frac{1}{2} \cdot \frac{5}{2} \cdot \frac{5}{2} \right]_0^2 \\
= \left[ \frac{2}{3} \left( \frac{3}{2} \right) - \frac{1}{2} \cdot \frac{5}{2} \cdot \frac{5}{2} \right] - (0) \\
= \frac{2}{3} \cdot 4\sqrt{2} - \frac{1}{5} \cdot 4\sqrt{2} = \frac{8\sqrt{2}}{15}
\]
(b) Find the exact length of the curve \( x = e^t + e^{-t}, \ y = 5 - 2t, \ 0 \leq t \leq 3. \)

\[
\frac{dx}{dt} = e^t - e^{-t}, \quad \frac{dy}{dt} = -2.
\]

\[
\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(e^t - e^{-t})^2 + (-2)^2} = \sqrt{e^{2t} + e^{-2t} - 4 + 4} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}.
\]

Thus

\[
\text{Length, } L = \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

\[
= \int_0^3 (e^t + e^{-t}) \, dt
\]

\[
= \left[ e^t - e^{-t} \right]_0^3 = e^3 - e^{-3}
\]
4. (20 points) Suppose a population $p(t)$ satisfies $\frac{dp}{dt} = 0.4p - 0.001p^2$, \( p(0) = 50 \) where \( t \) is measured in years.

(a) What is the carrying capacity?

(b) When will the population reach 50% of the carrying capacity?

\[ \frac{dp}{dt} = 0.4p \left(1 - \frac{0.01}{0.4} p\right) = 0.4p \left(1 - \frac{p}{400}\right). \]

Comparing the above differential equation with the logistic equation $\frac{dp}{dt} = kp \left(1 - \frac{p}{M}\right)$, we see that the carrying capacity $M = 400$.

(b) 50% of 400 = 200 and $A = M - P_0 = 400 - 50 = 350$.

We want to find $t$ such that

\[ p(t) = \frac{M}{1 + Ae^{-kt}} = \frac{400}{1 + 350e^{-0.4t}} = 200. \]

\[ 200 \left(1 + 350e^{-0.4t}\right) = 400. \]

\[ 1 + 350e^{-0.4t} = 2. \]

\[ e^{-0.4t} = 2 - \frac{1}{7} = \frac{1}{7}. \]

\[ t = \ln \left(\frac{1}{7}/0.4\right) \text{ years}. \]
5. (20 points) Sketch the curve $r = 2 \sin \theta$ and find the area it encloses in 2 ways:

(a) By using the formula for the area enclosed by a polar curve.

\[
\begin{array}{|c|c|}
\hline
\theta & r = 2 \sin \theta \\
\hline
0 & 0 \\
\frac{\pi}{2} & 2 \\
\pi & 0 \\
\frac{3\pi}{2} & -2 \\
2\pi & 0 \\
\hline
\end{array}
\]

\[
\text{Area} = \frac{1}{2} \int_0^\pi (2 \sin \theta)^2 \, d\theta = \frac{4}{2} \int_0^\pi \sin^2 \theta \, d\theta
\]

\[
= \frac{2}{2} \left[ \int_0^\pi (1 - \cos 2\theta) \, d\theta \right] = \left[ \theta - \frac{1}{2} \sin 2\theta \right]_0^\pi = 2\pi.
\]

(b) By finding a Cartesian equation for the curve then identifying the curve.

*Hint: Convert from polar to Cartesian coordinates using $\sin \theta = \frac{y}{r}$ and $r^2 = x^2 + y^2$ then complete squares to identify the curve.*

\[
\begin{align*}
r = 2 \sin \theta &= 2 \cdot \frac{y}{r} \\
\Rightarrow \quad r^2 &= 2y \\
x^2 + y^2 &= 2y \\
&\Rightarrow y^2 - 2y + x^2 = 0. \\
y^2 - 2y + (1)^2 - (1)^2 + x^2 = 0. \\
(y - 1)^2 + x^2 = 0 + 1 = 1.
\end{align*}
\]

Hence $r = 2 \sin \theta$ is a circle with radius 1 centered at $(0, 1)$.

Hence $\text{Area} = \pi r^2 = \pi \cdot 1^2 = \underline{\pi}$. 