Real Analysis Preliminary Exam

April 21, 2011

Write your codename, not your actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, Bluetooth, or other communication devices may be used during the exam.

Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are important. Do not make assumptions or choose contexts which make the problems trivial. If you use a theorem, state it fully and concisely, or identify it clearly. To receive full credit for a problem, the answer must be complete and correct. The scorers are not expected to supply any missing parts of any answer.

The weight of each problem is indicated to the left of the problem.

(12) **1.** Let $f \in L^1(\mathbb{R})$ and let $g(x) = \int_x^\infty f(t)\,dt$. Show that $g(x) \to 0$ as $x \to \infty$.

(12) **2.** Prove or disprove: If $f : [0,1] \to [0,1]$ is nondecreasing and continuous, then f is absolutely continuous.

(16) **3.** Let E be a closed Lebesgue measurable subset of $[0,1]$. Prove or disprove:

(8) **a.** If E^c is dense, then E has measure 0.

(8) **b.** If E has measure 0, then E^c is dense.

(12) **4.** Let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, and assume that $f' \in L^4([0,1])$ and that $f(0) = 0$. Find all values of α so that

$$\lim_{x \to 0+} x^{-\alpha} f(x) = 0$$

for all such f.

(16) **5.** Recall that, if (X,ρ) and (Y,σ) are metric spaces, a function $f : X \to Y$ is called Lipschitz continuous if there exists a constant $\lambda \geq 0$ such that

$$\sigma\big(f(x), f(y)\big) \leq \lambda\rho(x,y) \quad \text{for all} \quad x, y \in X\ .$$

Let X be a compact metric space, and let $Y = \mathbb{R}$, Show that the Lipschitz continuous functions are dense in the space of continuous functions with the uniform norm.

(16) **6.** Use an appropriate Fourier series to compute $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

(16) **7.** Assume that E is a compact Lebesgue measurable subset of $\mathbb{R}$, and let

$$f(t) = \int_E \cos(tx)\,dx, \quad t \in \mathbb{R}\ .$$

Prove or disprove:

(8) **a.** f has compact support.

(8) **b.** f is infinitely differentiable.