Manifolds & Topology

April 2011

This exam has two parts, A1, A2, A3, and B1, B2, B3. Please use separate blue books for each part A and B. Be sure to put your code name on each book and indicate clearly which problems are in each book. **Do not** write your real name on any book. Please explain your work clearly and indicate clearly what results you are using in your explanation. No calculators, books, cell phones, etc.

A1. i) Let X be a path connected & locally path connected space. Define a covering space

$$p : \tilde{X} \to X$$

ii) Let $p : \tilde{X} \to X$ be a covering space, $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$, $p(\tilde{x}_0) = x_0$. Describe the relation between this covering space and the fundamental groups of these spaces.

iii) Suppose $p : \tilde{X} \to X$ is a covering space with $p^{-1}(x)$ consisting of more than one point. Suppose $\tilde{X}$ is compact. Show that $\pi_1(X, x_0)$ contains a proper subgroup of finite index.

A2 i) State the Brouwer fixed point theorem.

ii) Let $D^2 = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$ $S^1 = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$, $X = D^2 - \{(0,0)\}$. Show that there is a continuous $f : X \to X$ with no fixed point.

iii) Is ii) still correct if X is replaced by D^2 with k interior points, $k > 1$, removed?

A3 i) Let Y be the plane $\mathbf{R}^2$ with k points removed, $k \geqq 0$. Show that the identity map $Id : S^2 \to S^2$, S^2 the 2-sphere, does not factor $S^2 \xrightarrow{f} Y \xrightarrow{2} S^2$ with f and g continuous. (Hint: if $k > 0$, Y has the homotopy type of a 1-dimensional simplicial complex.)

ii) Let S^{2n} be an even dimensional sphere, and suppose there was $f : S^{2n} \to S^{2n}$ continuous, $f(\vec{x}) \neq \vec{x}$, all $\vec{x} \in S^{2n}$. Show that f is then homotopic to the antipodal map. Conclude that **no** such f exists.

Manifolds Topology

B1. i) State the Whitney immersion and embedding theorems for a compact, smooth manifold M^n

ii) a) Show that there is a smooth immersion of the circle S^1 in the plane $\mathbf{R}^2$ which is not $1-1$.

b) Show that there is such an immersion of S^1 in $\mathbf{R}^2$ which is not $1-1$, but can be approximated, arbitrarily closely by an embedding if one considers $\mathbf{R}^2$ as a plane in $\mathbf{R}^3$.

B2. Let $\omega = (z - e^y)dx \wedge dy + (-x - 2y)dy \wedge dz + 2x(dx \wedge dz)$ be a differential 2 form in $\mathbf{R}^3$

a) Calculate $d\omega$

b) If $d\omega = 0$, find $\tau \ni d\tau = \omega$

B3. a) Let $T = S^1 \times S^1$. Suppose T is smoothly embedded in $\mathbf{R}^3$. Show that there are non-empty, open, subsets of T with Gaussian curvature positive and also such open sets with Gaussian curvature negative. (Hint: Use Gauss-Bonnet theorem)

b) Suppose T is an embedding as a circle of radius 1 around $(0, 0, 2) \in \mathbf{R}^3$ in the yz plane, which is rotated about the y-axis. Describe the points of Gaussian curvature zero.