

How to Control a Process to a Goal

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## Three problems

- Maximizing the probability of reaching a goal.
- Maximizing the probability of reaching a goal by a fixed time.
- Minimizing the expected time to reach a goal.

We first consider a famous example.

## Red-and-Black (Dubins and Savage, 1965)

You begin with fortune  $\$x$  and wish to reach a fortune of  $\$g$  where  $g > x$ . You may stake any amount  $s$ ,  $0 \leq s \leq x$  on an even money bet which you win with a fixed probability  $p$ . Your fortune at the next stage is then  $x_1$  where

$$x_1 = \begin{cases} x + s, & \text{with probability } p \\ x - s, & \text{with probability } 1 - p \end{cases}$$

You may then stake any amount  $s_1$ ,  $0 \leq s_1 \leq x_1$ , and so forth.

How should you choose your bets so as to maximize your chance of reaching the goal  $g$ ?

Let  $V(x)$  be your best chance of reaching the goal from  $x$ . What is the value of  $V(x)$ ?

## Bold Play

Bold play is the strategy that at every fortune  $x$ ,  $0 \leq x \leq g$  stakes the entire fortune  $x$  or just enough to reach the goal, whichever is smaller. Thus, for  $0 \leq x \leq g/2$ , the bold stake is  $x$  and the next fortune is

$$x_1 = \begin{cases} 2x, & \text{with probability } p \\ 0, & \text{with probability } 1 - p \end{cases}$$

If  $g/2 \leq x \leq g$  the stake is  $g - x$  and

$$x_1 = \begin{cases} g, & \text{with probability } p \\ x - (g - x) = 2x - g, & \text{with probability } 1 - p \end{cases}$$

**Theorem**(Dubins and Savage). If  $p \leq 1/2$ , then bold play is optimal.

There is no simple formula for the optimal return  $V(x)$ .

## **A generalization?**

**“Theorem”** (Coolidge, 1924). “The player’s best chance of winning a stated sum at an unfavorable game is to stake the sum that will bring that return in one turn. If that be not allowed, he should stake at each turn the largest amount the banker will accept.”

This “Theorem” applies to subfair red-and-black when there is a house limit on the size of bets. Unfortunately, it is false.

The error is repeated in the book “Gambling and Statistical Logic” by Epstein (2009).

## Red-and-Black with a House Limit

Suppose there is a limit  $L$  on the size of the bets allowed. If  $p \leq 1/2$ , it is natural to guess that it will be optimal to play as boldly as possible. The new version of bold play is then to stake at each fortune  $x$  the amount

$$s(x) = \min \{x, g - x, L\}.$$

Whether bold play is optimal now depends on whether the ratio  $L/g$  is or is not the reciprocal of an integer.

## Results for Red-and-Black with a House Limit

**Theorem** (Wilkins, 1972). If  $p \leq 1/2$  and  $L/g = 1/n$  for some positive integer  $n \geq 3$ , then bold play is optimal.

**Theorem** (Heath, Pruitt, & S, 1972). If  $\frac{1}{n+1} < \frac{L}{g} < \frac{1}{n}$  for some integer  $n \geq 3$  and  $p$  is sufficiently small, then bold play is not optimal.

**Theorem** (Schweinsberg, 2005). If  $p < 1/2$  and  $L/g$  is irrational, then bold play is not optimal.

## Superfair Red-and-Black on the Integers

Suppose that all fortunes and stakes are in integral numbers of dollars. So the possible stakes at fortune  $x$  are  $0, 1, \dots, x$ .

Assume that  $p > 1/2$ . Intuition now suggests that small bets will be good.

**Timid Play:** Stake  $s(x) = 1$  for  $1 \leq x \leq g - 1$ .

**Theorem**(Ross, 1974, Maitra and S.,1996). If  $p \geq 1/2$ , then timid play is optimal for red-and-black on the integers.

## The Return from Timid Play

Under timid play the process of fortunes forms a simple random walk

$$X_{n+1} = \begin{cases} X_n + 1, & \text{with probability } p \\ X_n - 1, & \text{with probability } 1 - p. \end{cases}$$

The return from timid play is

$$V(x) = P[\text{walk reaches } g \text{ before } 0].$$

The formula for  $V(x)$  is the classical “gambler’s ruin” formula:

$$V(x) = \frac{1 - \left(\frac{1-p}{p}\right)^x}{1 - \left(\frac{1-p}{p}\right)^g}$$

for  $x = 0, 1, 2, \dots, g$ .

## A word about proofs

Suppose that you have a guess as to the best strategy, say  $\sigma$ , and let  $Q(x) = P_\sigma[\text{reach } g \text{ from } x]$ .

Suppose further that you decide to first stake the amount  $s$  at  $x$  and then follow  $\sigma$ . Then your chance of reaching the goal will be

$$EQ(x_1) = p \cdot Q(x + s) + (1 - p) \cdot Q(x - s).$$

If  $\sigma$  is optimal, then

$$p \cdot Q(x + s) + (1 - p) \cdot Q(x - s) \leq Q(x).$$

Surprisingly, the converse is true; if this inequality holds for all  $x$  and  $s$ , then  $\sigma$  is optimal.

So we can prove optimality by checking this inequality.

## The proof that timid is optimal

Let  $Q(x)$  be the probability of reaching  $g$  playing timidly for  $x = 0, 1, \dots, g$ . Then, as already mentioned,

$$Q(x) = \frac{1 - \left(\frac{1-p}{p}\right)^x}{1 - \left(\frac{1-p}{p}\right)^g},$$

and we need to check that

$$p \cdot \frac{1 - \left(\frac{1-p}{p}\right)^{x+s}}{1 - \left(\frac{1-p}{p}\right)^g} + (1-p) \cdot \frac{1 - \left(\frac{1-p}{p}\right)^{x-s}}{1 - \left(\frac{1-p}{p}\right)^g} \leq \frac{1 - \left(\frac{1-p}{p}\right)^x}{1 - \left(\frac{1-p}{p}\right)^g},$$

for  $s = 0, 1, \dots, x$ . After easy algebra, this is

$$p\left(\frac{1-p}{p}\right)^s + (1-p)\left(\frac{1-p}{p}\right)^{-s} \geq 1,$$

which follows from the convexity of

$$s \rightarrow \left(\frac{1-p}{p}\right)^s.$$

## Superfair Red-and-Black on the reals

Suppose that your probability  $p$  of winning each bet is greater than  $1/2$  and that at  $x$  you can stake any  $s$  in  $[0, x]$ . Then you can reach the goal with probability one by staking a sufficiently small fixed proportion of your fortune at each stage.

**Theorem** If  $p > 1/2$ , then the strategy that stakes

$$s(x) = \alpha x$$

at each fortune  $x$  reaches the goal with probability 1 if  $\alpha$  is sufficiently small and positive.

## Sketch of the proof

Starting from  $x$  and staking  $\alpha x$ , your next fortune is

$$X_1 = \begin{cases} x + \alpha x & \text{with probability } p \\ x - \alpha x & \text{with probability } 1 - p \end{cases} = x(1 + \alpha Y_1)$$

where  $Y_1$  equals 1 or -1 with probability  $p$  or  $1 - p$  respectively. After  $n$  steps your fortune is

$$X_n = x \prod_{i=1}^n (1 + \alpha Y_i)$$

where  $Y_1, Y_2, \dots$  are IID. So

$$\log X_n = \log x + \sum_{i=1}^n \log(1 + \alpha Y_i) \rightarrow \infty \text{ a.s.}$$

if  $f(\alpha) = E[\log(1 + \alpha Y)] > 0$ .

Easy calculus shows  $f(\alpha) > 0$  for  $\alpha$  small and positive.

## Fair Red-and-Black

If  $p = 1/2$  then any strategy that is not “obviously bad” will be optimal. By “obviously bad” is meant a strategy that either bets so little that you stay forever in  $(0, g)$  with positive probability or bets so much that you overshoot the goal with positive probability.

Under a strategy that is not obviously bad the process  $x, X_1, X_2, \dots$  is a bounded martingale that eventually reaches either 0 or  $g$ . Let  $T$  be the time it first reaches one of these points. Then

$$\begin{aligned}x &= E(X_T) = g \cdot P[X_T = g] + 0 \cdot P[X_T = 0] \\ &= g \cdot P[X_T = g].\end{aligned}$$

So the goal is reached with probability  $x/g$  starting from  $x$ .

## A Continuous-Time Formulation

In red-and-black the process  $X_0 = x, X_1, X_2, \dots$  of fortunes can be described by its increments

$$\Delta X_n = X_{n+1} - X_n = s_n \cdot \Delta Y_n$$

where  $s_n$  is the  $n$ th stake and  $\Delta Y_n$  is the  $n$ th increment of a simple random walk. A continuous-time formulation is the stochastic differential

$$X_0 = x, \quad dX_t = s_t \cdot dY_t$$

where  $Y_t$  is a Brownian motion. If  $Y_t$  has drift  $\mu$  and standard deviation  $\sigma$ , then

$$dY_t = \mu \cdot dt + \sigma \cdot dW_t$$

where  $W_t$  is a standard Brownian motion.

## An Investment Problem

Let  $Y_t$  be the price of a certain stock at time  $t$  for each  $t \geq 0$  given by

$$dY_t = \mu \cdot dt + \sigma \cdot dW_t.$$

You begin with  $x \in (0, g)$ , your fortune is  $X_t$  at time  $t$ , and you may invest  $s_t$ ,  $0 \leq s_t \leq X_t$  in the stock. The process  $\{X_t, t \geq 0\}$  of your fortunes solves a stochastic differential equation

$$X_0 = x, \quad dX_t = s_t \cdot (\mu \cdot dt + \sigma \cdot dW_t).$$

How should you choose  $s_t$  to maximize your probability of reaching the goal  $g$ ?

Let  $V(x)$  be your best chance of reaching  $g$ . What is  $V(x)$ ?

**Theorem**(Pestien and S, 1985). If  $\mu \leq 0$ , the optimal strategy is “bold play” :

$$s_t = X_t, \quad \text{for all } t \geq 0.$$

The probability of reaching  $g$  is then

$$V(x) = (x/g)^{-2\mu+1}$$

If  $\mu > 0$ , you can reach the goal with probability one by taking

$$s_t = \alpha \cdot X_t, \quad \text{for all } t \geq 0,$$

for  $\alpha$  sufficiently small and positive.

The proof of the subfair case is, with the aid of stochastic calculus, much easier in continuous-time than is the Dubins and Savage result for discrete-time.

**Limited Playing Time.** Suppose you are playing red-and-black, but are limited to a fixed number, say  $n$ , of bets. How should you play to maximize your chances of reaching the goal?

**Subfair problems.** Bold play is fast as well as effective.

**Theorem**(Dvoretzky in Dubins and Savage). Bold play remains optimal in subfair ( $p \leq 1/2$ ) red-and-black with limited playing time.

**Superfair problems.** An optimal strategy must be time-dependent. Small bets are good when there is lots of time remaining, but big bets are necessary when time is running short. An exact solution for superfair ( $p \geq 1/2$ ) red-and-black is given by Kulldorf (1993).

## Minimizing the expected time to a goal

For superfair ( $p > 1/2$ ) red-and-black, you can reach the goal with probability one. What strategy gets there quickest?

**Kelly Criterion**(1956). Play at each stage to maximize  $E[\log X_n]$ .

Breiman (1961) showed this is “asymptotically optimal.” It cannot be optimal because of overshoot near the goal.

**Breiman’s Conjecture.** There is a number  $x^* \in (0, g)$  such that it is optimal to play Kelly for  $x < x^*$  and stake  $g - x$  for  $x \geq x^*$ .

There is a counterexample in Runyan (1998), who also shows that there is an optimal strategy.

**Open Question.** What is optimal?

## The Kelly Criterion in Continuous-Time

Recall that the stock price  $Y_t$  has stochastic differential

$$dY_t = \mu \cdot dt + \sigma \cdot dW_t.$$

If you invest  $\alpha X_t$  at every time  $t$ , then your fortune satisfies

$$X_0 = x, \quad dX_t = \alpha X_t \cdot (\mu \cdot dt + \sigma \cdot dW_t).$$

The process  $\{\log X_t\}$  has drift coefficient

$$\alpha\mu - \frac{\sigma^2\alpha^2}{2},$$

which is a maximum when  $\alpha = \mu/\sigma^2$ .

**Theorem**(Pestien & S, 1985). If  $\mu > 0$ , then the strategy which invests  $\mu/\sigma^2 \cdot X_t$  at every time  $t$  minimizes the expected time to reach  $g$ .

## Two-Person Red-and-Black

Assume now that there are two players, namely Mary and Tom. Mary begins with  $\$x$  and Tom begins with  $\$(g - x)$  where  $x$  and  $g$  are both integers with  $0 < x < g$ . Each player wants to win the other's money and end the game with  $\$g$ .

**Rules of the game.** When Mary has  $x$ , she can stake any amount  $a \in \{1, 2, \dots, x\}$ ; Tom with  $(g - x)$  can stake any amount  $b \in \{1, 2, \dots, g - x\}$ . There is a fixed probability  $p$  and Mary wins Tom's stake with probability

$$\frac{pa}{pa + (1 - p)b}$$

and Tom wins Mary's stake with probability

$$1 - \frac{pa}{pa + (1 - p)b} = \frac{(1 - p)b}{pa + (1 - p)b}.$$

At the next stage, Mary has fortune  $x_1$ , where

$$x_1 = \begin{cases} x + b, & \text{with probability } \frac{pa}{pa + (1-p)b} \\ x - a, & \text{with probability } \frac{(1-p)b}{pa + (1-p)b}. \end{cases}$$

Tom has fortune  $g - x_1$ .

Play continues in this fashion until one of the players reaches the goal  $g$ .

How should Mary and Tom play? To make sense of this question, we need a notion from game theory.

## Nash Equilibrium

In a game with  $n$  players, a Nash Equilibrium (NE) consists of a strategy for each player that is optimal for that player given that the other players play their given strategies.

A Nash Equilibrium need not be unique.

### Example: Sharing a cookie

Suppose Mary announces a proportion  $p_1$  and Tom announces a proportion  $p_2$ . If  $p_1 + p_2 \leq 1$ , then Mary receives her proportion  $p_1$  of the cookie and Tom receives his proportion  $p_2$ . However, if  $p_1 + p_2 > 1$ , then both players receive nothing.

For this game, the strategies  $p_1 = p$  and  $p_2 = 1 - p$  form a Nash equilibrium for every  $p$  in the interval  $[0, 1]$ .

## Two-Person Red-and-Black (continued)

Assume that  $p < 1/2$ . If Mary starts at  $x$  and stakes  $a$ , and Tom starts at  $g - x$  and stakes  $b$ , then

$$\begin{aligned} E(X_1) &= (x + b) \frac{pa}{pa + (1 - p)b} + (x - a) \frac{(1 - p)b}{pa + (1 - p)b} \\ &= x + \frac{(2p - 1)ab}{pa + (1 - p)b} \\ &< x. \end{aligned}$$

Thus the game is subfair for Mary and is superfair for Tom. This suggests Mary should make big bets and Tom should make small bets.

## **Nash Equilibrium for Two-Person Red-and-Black**

**Bold play for Mary:** When Mary has  $x$ , she stakes  $x$ .

**Timid play for Tom:** Tom always stakes 1.

**Theorem**(Pontiggia, 2005). These strategies form the unique Nash equilibrium.

## Three-Person Red-and-Black

Assume there are three players, Mary, Tom, and Jane. They begin with  $x, y,$  and  $z$  dollars, respectively, where

$$x + y + z = g.$$

Each player wants to win the total amount of money  $g$ .

**Rules of the game.** Mary stakes  $a \in \{1, 2, \dots, x\}$ , Tom stakes  $b \in \{1, 2, \dots, y\}$ , and Jane stakes  $c \in \{1, 2, \dots, z\}$ .

There is a fixed probability vector  $(p_1, p_2, p_3)$ . Mary, Tom, and Jane win the other players' stakes with probabilities

$$\frac{ap_1}{ap_1 + bp_2 + cp_3}, \quad \frac{bp_2}{ap_1 + bp_2 + cp_3}, \quad \frac{cp_3}{ap_1 + bp_2 + cp_3}.$$

Mary's next fortune is

$$x_1 = \begin{cases} x + b + c, & \text{with probability } \frac{ap_1}{ap_1 + bp_2 + cp_3} \\ x - a, & \text{with probability } 1 - \frac{ap_1}{ap_1 + bp_2 + cp_3}. \end{cases}$$

The next fortunes  $y_1$  and  $z_1$  for Tom and Jane are defined similarly.

Play continues until one of the players reaches  $g$ .

If one player, say Jane, reaches 0, the game becomes two-person red-and-black with the win probability for Mary equal to

$$\frac{ap_1}{ap_1 + bp_2}.$$

**Open question.** What is a Nash equilibrium?

## Some references

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