An Application of SOCP to Equity Margin Methodology

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September 10, 2010

Problem Statement

Some clients prefer their margin requirement to be stable with respect to modest changes in their portfolio and have a transparent calculation methodology. To this end, the business would like to offer clients a simple constraint set with the offer of a fixed margin charge (i.e., a fixed percent of the portfolio gross market value) so long as the client's portfolio satisfies the constraints. A core issue is the measurement of risk for a set of portfolios instead of a single portfolio, which we have addressed by attempting a worst case analysis. We hope to use this to

- determine simple constraints which may be used to control the risk of a long - short equity portfolio, and
- determine the sensitivity of that risk with respect to varying levels of the constraints.

Constraint Types and Risk Measurement

- Volatility
  - Portfolio level (correlation)
  - Security level
- Concentration
  - Security level
  - Sector level
  - Long - short skew

Risk is measured by looking at the worst 5-day return of a portfolio satisfying all the constraints. Five days is the holding period for a portfolio after a client has defaulted on their margin requirement and before we may seize and liquidate it. During this period we are exposed to market risk.

Notation

$w_i =$ weight of the i^{th} security in the portfolio on day 0,

$t_i =$ gross market value of the i^{th} security on day 0 ($= |w_i|$, heuristically)

$LMV_i =$ long market value of the i^{th} security on day 0 (i.e., $= 0.5(t_i + w_i)$,)

$SMV_i =$ short market value of the i^{th} security on day 0 (i.e., $= 0.5(t_i - w_i)$,)

$r_{i,k} =$ return of i^{th} security from day k to $k+1$,

$\alpha_{i,-k} = r_{i,-k} \prod_{j=1}^{k}(1 + r_{i,-j})^{-1}$

$\beta_{i,-k} = \prod_{j=1}^{k}(1 + r_{i,-j})^{-1}$

$R_i =$ return over a 5 day period $= \prod_{j=0}^{4}(1 + r_{i,j})$

Boundedness and Consistency Constraints

All the constraints under consideration are invariant with respect to scaling of the portfolio weights. For numerical stability, we impose a boundedness constraint,

$$\sum_{i=1}^{n} t_i = 1.$$

We cannot impose $t_i = |w_i|$ directly, as this is not a convex constraint. We relax this constraint as

$$\begin{aligned} w_i &\leq t_i, \\ -w_i &\leq t_i. \end{aligned}$$

Maximizing the risk tends to force one of these to become satisfied as equality, as we will see in the results below.

Portfolio volatility

Let GMV_k be the portfolio gross market value on day k and NMV_k be its corresponding net market value. The portfolio volatility is defined by

$$\begin{aligned} R_k &= \frac{NMV_{k+1} - NMV_k}{GMV_k}, \\ \overline{R} &= \frac{1}{N}\sum_{k=1}^{N} R_k, \\ \sigma_{Portfolio} &= \sqrt{\frac{1}{N}\sum_{k=1}^{N}(R_k - \overline{R})^2} \end{aligned}$$

In the original variables, we have

$$R_k = \frac{\sum_{i=1}^{n} \alpha_{i,-k} w_i}{\sum_{i=1}^{n} \beta_{i,-k} t_i}$$

and

$$\sigma_{Portfolio} = \sqrt{\frac{1}{N}\sum_{k=1}^{N}\left(\frac{\sum_{i=1}^{n} \alpha_{i,-k} w_i}{\sum_{i=1}^{n} \beta_{i,-k} t_i} - \frac{1}{N}\sum_{\acute{k}=1}^{N}\frac{\sum_{i=1}^{n} \alpha_{i,-\acute{k}} w_i}{\sum_{i=1}^{n} \beta_{i,-\acute{k}} t_i}\right)^2}$$

This function is not convex in $\{(w_i, t_i)\}_{i=1}^{n}$.

Implemented portfolio volatility constraint

Let Ω be the covariance matrix constructed using the weight adjusted returns $\alpha_{i,-k}$ instead of the actual returns $r_{i,-k}$. We fix the GMV of the portfolio to its current level and define the portfolio volatility as

$$\hat{\sigma}_{Portfolio} = \frac{\sqrt{w^t \Omega w}}{e^t t}$$

where $e = (1 \dots 1)^t$. Choosing L such that $L^t L = \Omega$, the constraint becomes

$$\|L^t w\| \leq \beta_{Vol} e^t t = \beta_{Vol}$$

Security level volatility constraints

To reduce exposure to correlation stability, we impose a constraint on the weighted average volatility of the portfolio. In addition, we also bound exposure to highly volatile stocks through our tail limits.

- Average volatility

$$\sum_{i=1}^{n} \sigma_i t_i \leq \beta_{AvgVol}$$

- Tail limits

$$\sum_{i=1}^{n} \chi_{\left\{\xi \geq \beta_{Vol_j}\right\}}(\sigma_i) t_i \leq W_j, j = 1 \ldots m.$$

Concentration constraints

- Security level

$$
\begin{aligned}
LMV_i \leq \beta_{LMV} \sum_{j=1}^{n} LMV_j \quad &\Leftrightarrow \quad t_i + w_i - \beta_{LMV} \sum_{j=1}^{n} (t_j + w_j) \leq 0 \\
&\Leftrightarrow \quad t_i + w_i - \beta_{LMV} \sum_{j=1}^{n} w_j \leq \beta_{LMV}, \\
SMV_i \leq \beta_{SMV} \sum_{j=1}^{n} SMV_j \quad &\Leftrightarrow \quad t_i - w_i - \beta_{SMV} \sum_{j=1}^{n} (t_j - w_j) \leq 0 \\
&\Leftrightarrow \quad t_i - w_i + \beta_{SMV} \sum_{j=1}^{n} w_j \leq \beta_{SMV}.
\end{aligned}
$$

More concentration constraints

- Sector level
  Let $\Psi = \{s | s \text{ is an industry sector}\}$ and let $I_s = \left\{ i \in \{1 \dots n\} | \text{the issuer of the } i^{th} \text{ security is in sector } s \right\}$. Then

$$\forall s \in \Psi, \sum_{i \in I_s} t_i \leq \beta_{Sector}.$$

- Long - short skew

$$\sum_{j=1}^{n} LMV_j \leq \beta_{Skew} \sum_{j=1}^{n} SMV_j \Leftrightarrow \sum_{j=1}^{n} t_j + w_j \leq \beta_{Skew} \sum_{j=1}^{n} t_j - w_j$$

$$\Leftrightarrow \sum_{j=1}^{n} w_j \leq \frac{\beta_{Skew} - 1}{\beta_{Skew} + 1},$$

$$\sum_{j=1}^{n} SMV_j \leq \beta_{Skew} \sum_{j=1}^{n} LMV_j \Leftrightarrow \sum_{j=1}^{n} t_j - w_j \leq \beta_{Skew} \sum_{j=1}^{n} t_j + w_j$$

$$\Leftrightarrow \sum_{j=1}^{n} w_j \geq \frac{1 - \beta_{Skew}}{\beta_{Skew} + 1}.$$

Objective

We search among all portfolios satisfying the constraints those which have the worst loss over the ensuing 5 days; i.e.,

$$\text{minimize } R^t w$$

The linear objective and linear and quadratic constraints result in this being a second order cone program and efficiently solvable through interior point methods. We have implemented our own primal-dual solver capable of handling large sparse constraint matrices. It is based on an algorithm presented in (Boyd, et. al.) and modified appropriately to handle conic constraints.

Test description

We solved for worst case portfolios over the period November 5^{th}, 2007 to June 25^{th}, 2010, totaling 690 data sets per choice of parameters. In our 8 test cases, we held the following values for parameters below:

$$\begin{aligned} \beta_{AvgVol} &= 60\%, \\ W_1 &= 10\%, \\ W_2 &= 20\%, \\ \beta_{Sector} &= 10\%. \end{aligned}$$

Sector classification was based on GICs Level 2 (24 industrial classes). We took $\beta_{LMV} \equiv \beta_{SMV} =: \beta_{IC}$ and varied $\beta_{Vol}, \beta_{Vol_1}, \beta_{Vol_2}, \beta_{IC}$, and β_{Skew} as indicated in the following chart. Optimizations were considered successful if $\sum_{i=1}^{n} |w_i| \geq 0.95 \sum_{i=1}^{n} t_i$.

Computational results

# succesful optimizations	β_{Vol}	β_{Vol_1}	β_{Vol_2}	β_{IC}	β_{Skew}	Average Return	Std Dev	99% quantile
678	30%	100%	80%	5%	300%	-11.89%	3.91%	-24.74%
654	20%	100%	80%	5%	300%	-11.45%	3.51%	-23.48%
687	30%	90%	70%	5%	300%	-11.81%	3.84%	-24.40%
668	20%	90%	70%	5%	300%	-11.41%	3.45%	-22.96%
660	30%	100%	80%	2.5%	300%	-9.92%	3.47%	-21.57%
639	20%	100%	80%	2.5%	300%	-9.60%	3.15%	-20.04%
666	30%	90%	70%	2.5%	300%	-9.86%	3.41%	-21.76%
656	20%	90%	70%	2.5%	300%	-9.56%	3.07%	-19.73%
677	30%	100%	80%	5%	200%	-11.51%	3.81%	-23.94%
659	20%	100%	80%	5%	200%	-11.20%	3.47%	-23.23%
686	30%	90%	70%	5%	200%	-11.42%	3.70%	-23.60%
674	20%	90%	70%	5%	200%	-11.16%	3.41%	-22.36%

Table: Optimization results for the period November 5^{th}, 2007 to June 25^{th}, 2010.

Conclusion

Our current analysis implies leverage bounds of 4 - 1 or 5 - 1. The current client offering is 10 - 1. We need to decide whether the analysis is too conservative in its use of the worst case portfolio or whether the risk is real and the business chooses to accept this risk to be competitive in this market.

Boyd, Stephen and Vendenberghe, *Convex Optimization* (Cambridge: Cambridge University Press, 2006)

Wright, Stephen J., *Primal-Dual Interior-Point Methods* (Philadelphia: SIAM, 1997)

Quantitative Prime Services

Quantitative support for Prime Services has traditionally focused on issues concerning risk, margin strategies, and business development. Most analysis relies heavily on historical data and focuses on "real world" as opposed to risk neutral returns. Our current focus is to attract more clients by offering margin methodologies suitable for their trading strategies.

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