# Quantile Curiosa

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# Introduction

- The distribution function (df) of a random variable X is defined as: $F_X(x) = \Pr[\, X \le x \,]$
- The generalized inverse for a df, the quantile function $q_X(p)$, is defined as:

  $$F_X^{-1}(p) = \inf \{ \text{real } x \mid F_X(x) \ge p \}$$
  $$= \sup \{ \text{real } x \mid F_X(x) < p \}$$
- The 'check' function is defined as:

  $\rho_q(r) = qr - r1_{\{r < 0\}}$ for $0 \le q \le 1$,

  e.g. $\rho_{0.5}(r) = 0.5|r|$

# Curious folk result

For an integrable rv X, the minimizer of $E[\rho_q(X - x)]$ with respect to x, is the q-quantile of X.

For an elementary proof, see [Hunter, Lange 1998, Appendix]

# An even more curious result

Suppose rv X has $E[\,|X|\,] < \infty$. Then the Fenchel-Legendre transform of the convex function $\Psi(x) = E[\,(x - X)^+\,]$ is given by

$\Psi^*(y) = \sup_{x \,\varepsilon\, \text{Real}} (\, xy - \Psi(x)\,)$

= Integral from 0 to y of $q_X$, if $0 \le y \le 1$,

and $+\infty$ otherwise.

Moreover, for $0 < y < 1$, the supremum above is attained in x if and only if x is a y-quantile of X, that is $x = q_X(y)$.

[Follmer, Schied, 2004, Lemma A.22]

# Their connection!

$\arg\min_{x \varepsilon R} \{E\,[\rho_q\,( X - x)\,]\,\}$

$= \arg\min_{x \varepsilon R} \{\, qE[X] - qx - E[\,(X - x)1_{\{X-x < 0\}}\,]\,\}$

$= \arg\max_{x \varepsilon R} \{\, qx + E[\,(X - x)1_{\{X-x < 0\}}\,]\,\}$

$= \arg\max_{x \varepsilon R} \{\, xq - E[\,(x - X)^+\,]\,\}$

$= \arg\max_{x \varepsilon R} \{\, xq - \Psi(x)\,\}$

$= q_X(q)$, for $0 < q < 1$.

So we have an elegant proof of
the folk result via functional analysis!

# Applications

Quantile regression is a statistical technique used to estimate and make inference about conditional quantile functions [Koenker, Bassett, 1978]. Financial applications of quantile functions include

asset pricing [Follmer, Schied, 2004],
portfolio construction [Ma, Pohlman, 2005], [Bassett, Koenker, Kordas, 2004],

risk management [Chernozhukov, 2002], [McNeil, Frey, Embrechts, 2005], and

insurance [Denuit, Dhaene, Goovaerts, Kaas, 2005].

# Some quantile function properties

First order quantile ODE:

$dq/dp = 1/f(q)$ where $q$ is the quantile function, $0 \le p \le 1$, and $f$ is the pdf.

Second order non-linear ODE:

$d^2q/dp^2 = H(q)\,(dq/dp)^2$

where $H(q) = -d/dq \ln\{ f(q) \}$.

For power series solutions, see [Steinbrecher, Shaw, 2007].

The quantile-characteristic function connection:

$\phi_X(t) := E[\exp(itX)]$ = Integral from 0 to 1 of $\exp(itq_X)$

is explored via differentiation in [Shaw, McCabe, 2009].

May you discover more curious quantile properties!

# References


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