A Study of Equivalance in CVaR Portfolio Optimization

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”The special sphere of finance within economics is the study of the allocation and deployment of economic resources...in an uncertain environment.”

Milton Friedman

”It is not at all that laws of nature exist, much less that man is able to discover them.”

Erwin Schrodinger

The benchmark mathematical formulation of portfolio risk and reward is given by,

$$\begin{aligned} \text{minimize} \quad & \mathbf{w}'\Sigma\mathbf{w} \\ \text{subject to} \quad & \mu'\mathbf{w} \geqslant \alpha \\ & \mathbf{1}'\mathbf{w} = 1 \\ & \mathbf{w} \geqslant 0, \end{aligned}$$

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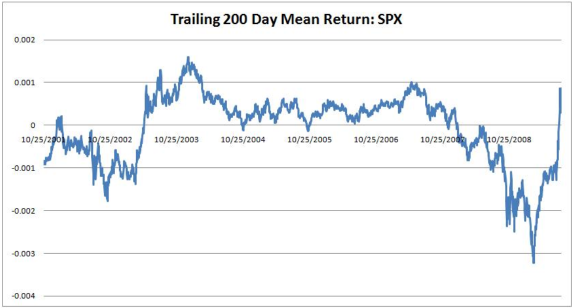

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- Every investor has the same utility over a fixed horizon
- That utility is quadratic in risk; viz., variance
- This necessitates (or is justified by) a geometric brownian motion for the underlying assets

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Motivated by the last statement, we may examine the effect of constraints on an MVO problem. Jagannathan and Ma note (under some conditions)

- Lower bounds adjust the mean upward proportional to the Lagrange multiplier
- Upper bounds adjust the mean downward proportional to the Lagrange multiplier
- A nonnegativity constraint is equivalent to shrinking Σ towards a single factor covariance matrix.

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$$f(w, y) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$$

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Assuming that the scenarios have probability density function *p*, the cumulative distribution function of losses, given portfolio weights *w*, is

$$\Psi(x, \gamma) = \int_{f(x,y)<\gamma} p(y)dy$$

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$$\mathrm{VaR}_{\alpha}(w) = \min\{\gamma \in \mathbb{R} \,|\, \Psi(w, \gamma) \geqslant \alpha\}$$

We have that $\mathrm{VaR}_{\alpha}(w)$ is the smallest amount of loss that we can expect with probability $1 - \alpha$

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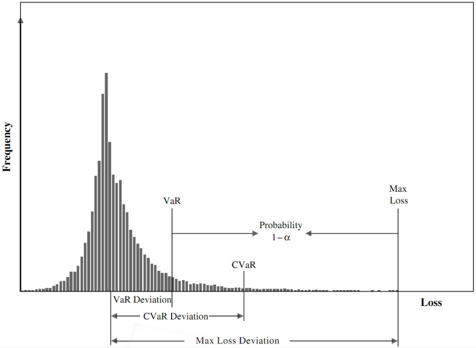

We can discretize this in a natural way by sampling our scenarios discretely according to p (see Rockafeller (1999)), and write another optimization problem:

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A linear programming problem.

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We will be mainly interested in

- Examining the impact of simple constraints on a CVaR optimal portfolio and the underlying joint density
- Investigating
- Out of sample performance

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