Transitivity: $k \to A \to B$, M a B-module.

Then get long exact sequence

$$\cdots H^{s+1}(B/A, M) \leftarrow H^s(A/k, M) \leftarrow H^s(B/k, M) \leftarrow H^s(B/A, M) \leftarrow \cdots$$

16.10.03

Announcement: Γ-cohomology $\overset{\cong}{\longleftrightarrow} H^*_{\mathcal{E}(s\mathcal{M}od_k)}(-)$

follows from work of Basterra–McCarthy, Mandell.

§1 Adams type spectra

E: homotopy commutative ring spectrum

(1) $E = \operatorname{hocolim}_\alpha E_\alpha$ with E_α finite CW (filtered)

(2) $E_* DE_\alpha$ is a projective E_*-module (D = Spanier-Whitehead dual)

(3) $[DE_\alpha, F] \longrightarrow \operatorname{Hom}_{E_*E}(E_* DE_\alpha, E_*F)$ is an iso

$\xrightarrow{\cong} \operatorname{Hom}_{E_*}(E_* DE_\alpha, F_*)$ for all E-module spectra F.

All Landweber exact spectra satisfy these.

Lemma: If $[\Sigma^n DE_\alpha, X] \xrightarrow{\cong} [\Sigma^n DE_\alpha, Y]$ for $f: X \to Y$ and all n,

then $E_*f: E_*X \to E_*Y$ is also iso.

Pf: $E_n X = [S^n, E \wedge X] = \operatorname{colim}_\alpha [S^n, E_\alpha \wedge X]$

$= \operatorname{colim}_\alpha [\Sigma^n DE_\alpha, X]$

§2. <u>Stover resolutions</u> (Bousfield)

Let $\mathcal{P} = \{\Sigma^n DE_\alpha\}_{n,\alpha}$

- A map $f: X \to Y$ of spectra is <u>$\mathcal{P}$-epi</u> if

$$[P,X] \to [P,Y] \text{ for all } P \in \mathcal{P}.$$

- A spectrum Q is $\mathcal{P}$-projective if $[Q,X] \xrightarrow{f_*} [Q,Y]$ is epi for all $\mathcal{P}$-epis $f: X \to Y$

Note: $\underline{\underline{S}}$ has enough $\mathcal{P}$-projectives, namely

$$\left(\bigvee_{P \in \mathcal{P}} \bigvee_{f: P \to X} P \right) \longrightarrow X .$$

- $f: A \to B$ is a $\mathcal{P}$-projective cofibration if it has the LLP

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "\text{fibration and } \mathcal{P}\text{-epi}"] \\ B \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

<u>Thm</u> (Dwyer-Kan-Stover, Goerss-Hopkins, Bousfield, Hirschhorn)

$s\underline{\underline{S}}$ is a simplicial model category where

(i) weak equivalences $= E_*$-equivalences $=$ $f: X_\bullet \to Y_\bullet$ such that $\pi_* E_* X \to \pi_* E_* Y$ is iso.

(ii) f is E_*-cofibration $\iff$

$$X_n \cup_{L_n X} L_n Y \longrightarrow Y_n \text{ is a } \mathcal{P}\text{-proj. cofibration.}$$

$$L_n X = \text{“degeneracy”} = \text{latching object} = \operatorname*{colim}_{\substack{\varphi: [n] \twoheadrightarrow [k] \\ k<n}} X_k .$$

This is a localization of the "E_2-model structure" in which weak equivalences are those $f: X \to Y$ which induce isos on

$$\pi_* [\Sigma^n DE_\alpha, X_\bullet] \to \pi_* [\Sigma^n DE_\alpha, Y_\bullet] \text{ for all } \alpha, n$$

Lemma: (Bousfield) $f : X \to Y$ is a $\mathcal{P}$-cofibration iff f is a retract of

$$X \xrightarrow{i} X \vee F \xrightarrow[\substack{\text{cofibration} \\ \text{(weak equiv. in } \underline{\underline{S}})}]{\text{acyclic}} Z \quad \text{with } F \ \mathcal{P}\text{-projective.}$$

Note: If Q is $\mathcal{P}$-projective, then $E_* Q$ is E_*-projective.

Notation: If $X \in \underline{\underline{S}}$ is regarded as constant simplicial spectrum, then a Stover resolution is a cofibrant approximation

$$s\underline{\underline{S}} \ni Y_\bullet \xrightarrow{\sim_{E_*}} X \quad \text{in the above model structure.}$$

Example: If $Y_\bullet \in s\underline{\underline{S}}$ is E_*-cofibrant, then $E_* Y_\bullet \in s\mathrm{Mod}_{E_*}$ is cofibrant.

Thm: Let $\mathcal{C} = s\mathrm{Alg}_{A_\infty}$ or $\mathcal{E}(s\underline{\underline{S}})$. Then

$$\text{forget} : \mathcal{C} \longrightarrow s\underline{\underline{S}} \quad \text{creates an } E_*\text{-model structure.}$$

Moreover, if $X_\bullet \in \mathcal{C}$ is E_*-cofibrant, then

$$E_* X_\bullet \in \left\{ \begin{matrix} s\mathrm{Alg}_{E_*} \\ \text{or} \\ \mathcal{E}(s\mathrm{Mod}_{E_*}) \end{matrix} \right\} \text{ is cofibrant.}$$

§3. Mapping and moduli spaces

The realization functor $|\bullet| : \mathcal{E}(s\underline{S}) \longrightarrow \mathrm{Alg}_{E_\infty}$ preserves weak equivalences between cofibrant objects.

Proposition: Let X, Y be E_∞-algebras, regarded as constant objects in $\mathcal{E}(s\underline{S})$

1) $\mathrm{map}_{\mathcal{E}(s\underline{S})}(X, Y) \simeq \mathrm{map}_{E_\infty}(X, Y)$ derived mapping spaces

DNTH

2) Let $\mathcal{R}_\infty(A)$ denote the realization category of all $Y_\bullet \in \mathcal{E}(s\underline{S})$ such that

$$\pi_0 E_* Y \cong A \quad \text{as } E_*E\text{-comodule algebras}$$
$$\pi_k E_* Y = 0 \quad \text{for } k>0.$$

Then $B\mathcal{R}_\infty(A) \simeq B\mathcal{R}(A)$.

Proof:

$$\begin{tikzcd}
\mathcal{R}^{cf}_\infty(A) \arrow[r, shift left, "\simeq"] \arrow[d, "1-1"] & \mathcal{R}_\infty(A) \arrow[l, shift left, "\text{funct. cofibrant replacement}"] \\
\mathcal{R}(A) \arrow[u, bend left] \arrow[ur, "\text{constant object}"'] &
\end{tikzcd}$$

$\square$

§4. Postnikov towers

For $P \in \mathcal{P}$, ~~define~~ and $X_\bullet \in s\underline{S}$, $\mathcal{E}(s\underline{S})$ etc

$$\pi_n^\natural(X_\bullet, P) = \pi_{P,n}(X_\bullet) := \pi_n \operatorname{map}_{s\underline{S}}(P, X_\bullet)$$
$$= \left[\frac{P\otimes\Delta^n}{P\otimes\partial\Delta^n}, X_\bullet\right]_{s\underline{S}}$$

We have $\pi_{P,0}(X_\bullet) = \pi_0[P, X_\bullet]$ and there is a long exact sequence

$$\pi_2[P,X_\bullet] \longrightarrow \pi_{\Sigma P,0}(X_\bullet) \longrightarrow \pi_{P,1}(X_\bullet) \longrightarrow \pi_1[P,X_\bullet] \longrightarrow 0$$

Set $\pi_{E_*,n}(X_\bullet) = \operatorname{colim}_\alpha \pi_{\Sigma^k DE_\alpha, n}(X_\bullet)$

Then $\pi_{E_*,*}(X_\bullet) \longrightarrow \pi_{E_*,*}(Y_\bullet)$ is iso

$\Leftrightarrow \quad \pi_* E_* X_\bullet \longrightarrow \pi_* E_* Y_\bullet$ is iso

Example: If $\pi_* E_* X_\bullet \cong A$ is degree 0, then

$$\pi_{E_*, n}(X_\bullet) \cong A[n] \cong \Sigma^{-n} A$$

same grading.

For $X_\bullet \in s\underline{\underline{S}}$ there is a tower under $X_\bullet$

$$X_\bullet \longrightarrow \cdots \longrightarrow P_2 X_\bullet \rightarrow P_1 X_\bullet \rightarrow P_0 X_\bullet$$

such that

$$\pi_{P,k} P_n X_\bullet \cong \begin{cases} 0 & \text{for } k > n \\ \pi_{P,k} X_\bullet & \text{for } k \leq n. \end{cases}$$

The Layers: If A is an $E_* E$ comodule algebra, then $B_A \in \mathcal{E}(s\underline{\underline{S}})$ is defined by the property

$$\pi_0 \operatorname{map}_{\mathcal{E}(s\underline{\underline{S}})}(X_\bullet B_A) \cong \operatorname{Hom}_{E_*E\text{-alg}}(\pi_0 E_* X, A)$$

Prop: B_A exists, is essentially unique and satisfies

$$P_0 X_\bullet = B_{\pi_0 E_* X}.$$

If M is an A-module, then a map $B_A \longrightarrow B_A(M, n)$ is of type (M, n) $(n \geq 1)$ if

1. $\pi_i E_* B_A \longrightarrow \pi_i E_* B_A(M, n)$ is iso for $i < n$
2. $\pi_n E_* B_A(M, n) = M$ as an A-module
3. $\pi_{P,k} B_A(M, n) = 0$ for $k > n$.

"twisted EM-object"

The conditions imply that $P_{n-1} B_A(M, n) \cong B_A$

Prop: 1) $B_A(M,n)$ exist.

2) For $Y_\bullet \in \mathcal{E}(s\mathcal{A}_g)$ and $f: \pi_0 E_* Y_\bullet \to A$ given,

$$[Y_\bullet, B_A(M,n)]_{\mathcal{E}(s\underline{S})/B_A} \cong H^n_{\mathcal{E}(s\mathcal{M}od_{E_*E})}(E_*Y_\bullet/A, M)$$

$\Rightarrow$ AQ cohomology is representable.

$$\begin{array}{ccc} P_n X_\bullet & \longrightarrow & P_0 X_\bullet = B_{\pi_0 E_* X_\bullet} \\ \downarrow & & \downarrow \\ P_{n+1} X_\bullet & \longrightarrow & P_{n+1} \underset{\text{in } \mathcal{E}(s\underline{S})}{(\text{pushout})} \simeq B_{\pi_0 E_* X_\bullet}(\pi_{E_*,n} X, n+1) \\ & \uparrow & \end{array}$$

k-invariant, lives in AQ-cohomology.

The realization space

Def: Let A be an E_*E-comodule algebra. Then $0 \le n \le \infty$ $X \in \mathcal{E}(s\underline{S})$ is a potential n-stage for A if

① $\pi_0 E_* X \cong A$

② $\pi_i E_* X_\bullet = 0$ for $1 \le i \le n+1$

③ $\pi_{P,k}(X_\bullet) = 0$ for $k > n$.

The conditions apply that $\pi_k E_* X_\bullet = \begin{cases} A & \text{for } k=0 \\ A[n] & \text{for } k=n+2. \end{cases}$

Let $\mathcal{R}_n(A)$ be the category of potential n-stages for A with E_*-equivalences.

Then we get a tower

$$BR_\infty(A) \to \cdots \to BR_n(A) \xrightarrow{p_{n-1}} BR_{n-1}(A) \to \cdots \to BR_0(A)$$
$$\begin{matrix} \| \\ BR(A). \end{matrix}$$

<u>Theorem</u> (Dwyer-Kan)

$$BR_\infty(A) \simeq \operatorname{holim}_n BR_n(A)$$

<u>Theorem</u>: $BR_0(A) = B\mathrm{Aut}_{E_*E\text{-alg}}(A)$ (connected, non-empty !)

and there is a pullback diagram

$$\begin{tikzcd}
BR_n(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B\Gamma \arrow[d] \\
BR_{n-1}(A) \arrow[r] & E\Gamma \times_\Gamma \mathcal{H}^{n+2}_{E(sMod_{E_*E})}(A/E_*, A[n])
\end{tikzcd}$$

$\Gamma = \mathrm{Aut}(A, A[n])$ automorphisms of the pair (algebra, module).