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Moduli spaces and obstruction theory

§1 : The basic moduli problem

Let E_* be a homology theory based on a ring spectrum
so that $\quad \eta_L : E_* \to E_*E$ is flat ($\Rightarrow$ good theory of comodules)

Let A be a commutative algebra in E_*E-comodule ("comodule algebra")
Find all E_∞ ring spectra X so that $\quad E_*X \cong A$.

Category $\mathcal{R}(A)$ "realization category"

$\mathrm{ob}\,\mathcal{R}(A) = X \in \mathrm{Alg}_{E_\infty}$ so that $E_*X \cong A$ (not part of data)
$\mathrm{mor}\,\mathcal{R}(A) = (X \to Y) \in \mathrm{Alg}_{E_\infty}$ so that E_*f is iso.

Moduli space for this problem: $\quad B\mathcal{R}(A) = $ nerve of $\mathcal{R}(A)$.

Basic problem: calculate the homotopy type of $B\mathcal{R}(A)$ if $B\mathcal{R}(A) \neq \emptyset$.

If $\quad B\mathcal{R}(A) \neq 0 \iff \exists X$ such that $E_*X \cong A$
Then: $\quad \pi_0 B\mathcal{R}(A) = E_*$-iso classes of X's.

Thm (Dwyer-Kan):

$$B\mathcal{R}(A) \simeq \coprod_{[X] \in \pi_0} B\,h\mathrm{Aut}_{E_\infty}(X)$$

with $h\mathrm{Aut}_{E_\infty}(X)$ space of self E_*-equivalences of some fibrant/cofibrant X in $[X]$.

§2 Spectra and E_∞ ring spectra

$\underline{S}$ = a category of spectra

Axiom 1 $\quad \mathrm{Ho}(\underline{S})$ = stable homotopy category

$\underline{S}$ is a cofibrantly generated simplicial model category Quillen equivalent to Bousfield–Friedlander spectra. Technical: generators should have cofibrant source.

Axiom 2. $\underline{S}$ has closed symmetric monoidal smash product which descends to the usual smash product on homotopy category.

Axiom 3: If S is the sphere spectrum and K a space, then

$$K \otimes X \cong (K \otimes S) \wedge X \cong \quad \text{(i.e., 2 \& 3 are compatible)}$$

Axiom 4: If $K \to L$ is a morphism of Σ_n-spaces which is a weak equivalence as spaces, and if X is a cofibrant spectrum, then

$$K \otimes_{\Sigma_n} (\underbrace{X \wedge \cdots \wedge X}_{n}) \longrightarrow L \otimes_{\Sigma_n} (\underbrace{X \wedge \cdots \wedge X}_{n}) \text{ is a weak equivalence.}$$

Theorem: Such $\underline{S}$ exist.

Examples: EKMM S-modules, MMSS symmetric spectra on top. spaces in positive model structure,

HSS symmetric spectra on simplicial sets.

Example: If $E\Sigma_n$ is your favorite free contractible Σ_n-space, then $E\Sigma_n \to *$ satisfies hypotheses of axiom 4. So

$$E\Sigma_n \otimes_{\Sigma_n} X^{\wedge n} \longrightarrow X^{\wedge n}/\Sigma_n \quad \text{is a weak equivalence}$$

for cofibrant X.

Definition: E_∞ ring spectra := category of commutative monoids in $\underline{S}$ under $\wedge$.

$$= Alg_{E_\infty}.$$

A_∞-ring spectra := cat. of associative monoids $= Alg_{A_\infty}$

Thm: Alg_{E_∞} is Quillen equivalent to algebras over any E_∞-operad.

Axiom 5: For any operad of simplicial sets, the category $Alg_{\mathcal{C}}$ inherits a model category structure (i.e. fibrations and weak equivalences defined in $\underline{S}$).

§3 André–Quillen cohomology.

Suppose $X \to Y$ is a morphism of E_∞ ring spectra. Want to calculate

$$\pi_*\left(\mathrm{map}_{E_\infty}(X, Y; \mathcal{J})\right)$$

$$\pi_t\, \mathrm{map}_E(X,Y) \cong [X, Y^{s^t}]_{Alg_{E_\infty}/Y} \longrightarrow \mathrm{Hom}_{E_* alg/E_*}(E_*X, E_*(Y^{s^t}))$$

Here $E_*(Y^{s^t}) \cong E_*Y \oplus \Sigma^{-t}E_*Y = E_*Y[\varepsilon_{-t}]$, $\varepsilon_{-t}^2 = 0$

André–Quillen cohomology is the derived functor of this.

If A is a k-algebra, M an A-module, the "square zero extension" is

$$A \ltimes M = A \oplus M \text{ with product}$$

$$(a,x)(b,y) = (ab, xb + ay)$$

If $K(A,n)$ is the ~~k, A, M have~~ simplicial ~~directions~~ A-module whose normalization is $M[n]$

$$A \ltimes K(M,n) \in sAlg_k .$$

Then

$$\mathcal{H}^n(A/k; M) = \mathrm{map}_{sAlg_k/A}(A, A \ltimes K(M,n))$$

Here and in following all spaces of maps are <u>derived</u>; replace source and target by cofibrant resp. fibrant objects as necessary.

Then

$$H^n(A/k; M) = \pi_0 \mathcal{H}^n(A; M) \cong \pi_k \mathcal{H}^{n+k}(A; M).$$

Note: $E_*(Y^{s^t}) = E_*Y \ltimes (\Sigma^{-t}E_*Y)$ is an object of E_*E-comodule algebra.

If A is E_*-comodule algebra and M an A-module in E_*-comodules, then define similarly

$$\mathcal{H}^n_{E_*E}(A/E_*; M)$$

If $M = E_*E \otimes_{E_*} M_0$ is an extended comodule, then

$$\mathcal{H}^n_{E_*E}(A/E_*, E_*E \otimes_{E_*} M_0) \simeq \mathcal{H}^n(A/E_*, M_0)$$

Example: $M = E_* E$.

Remark: $\mathcal{HC}^n(A; M)$ depends very much on the ground category, which is suppressed from notation:

- associative algebras
- commutative algebra (does not apply, since E_*(E_∞-algebra) has extra structure)

One needs that $E_*(\text{free } E_\infty(X)) = $ some functor of $E_* X$.

Note: for decent E_*, $E_*(\text{Tensor}(X)) = \text{Tensor}(E_* X)$ if $E_* X$ is flat as E_*-module.

Consider Alg_{A_∞} with E_*-isomorphisms as weak equivalences. (Localisation of Axiom 5 model structure).

Thm: Let $\phi : X \to Y$ be a morphism in Alg_{A_∞}. Then there is a spectral sequence

$$E_2^{s,t} \Longrightarrow \pi_{t-s}(\mathrm{Alg}_{A_\infty}(X, Y_E), \phi)$$

with $E_2^{0,0} = \mathrm{Hom}_{E_* E\text{-alg}}(E_* X, E_* Y)$ and

$$E_2^{s,t} = H^s_{\mathrm{assoc}}(E_* X / E_*, \Sigma^{-t} E_* Y)$$

where $Y_E = E$-completion of Y.

Bousfield: given algebraic map $f : E_* X \to E_* Y$, there are obstructions

$$\Theta_s \in H^{s+1}_{E_* E\text{-}\mathrm{assoc}}(E_* X / E_*; \Sigma^{-s} E_* Y)$$

to realising f as a map of A_∞-algebras $X \to Y$. (obstructions lie in "(-1)-stem").

Theorem : There exist successively defined obstructions
$$\xi_s \in H^{s+2}_{E_*E,\,assoc}(E_*A|_{E_*}, E_*X[s])$$ "(-2)-stem"
to realizing A as an A_∞-algebra
(here $(M[t])_n = M_{t+n}$, so that $M[t] = \Sigma^{-t}M$)

Having such a realization gives a spectral sequence
$$H^s_{E_*E\text{-assoc}}(E_*X|_{E_*}, E_*X[t]) \Rightarrow \pi_{t-s+1} BR(A)$$

14.10.03

Recall : We defined AQ-cohomology object
$$H^n_{assoc}(A/k; M) = \pi_0 \operatorname{map}_{k\text{-alg}/A}(A, A \ltimes K(M,n))$$
$$\operatorname{ob}(k\text{-alg}/A) = \{ B \xrightarrow{\text{alg}} A \}$$

There was a respective E_*E-comodule version, and the AQ-cohomology groups came up in obstruction theory.

AQ-cohomology for E_∞-algebras

To form the derived space of maps, need to take a cofibrant replacement $X_\bullet \to A$ in simplicial associative algebras. Thus
$$\pi_n X_\bullet = \begin{cases} 0 & \text{for } n \neq 0 \\ A & \text{for } n = 0. \end{cases}$$
Cofibrant: forgetting the face maps, $X_\bullet \cong \operatorname{Tensor}_*(M_\bullet)$ for some simplicial k-module which is projective in each dimension.

If $X \in \operatorname{Alg}_{A_\infty}$, we can imagine a simplicial A_∞-ring spectrum $Z_\bullet \to Y$ such that, forgetting the face maps $Z_\bullet \cong T(M_\bullet)$ such that $E_*(M_n)$ is a projective E_*-module. free A_∞

Then $E_* Z_\bullet \cong E_* T(M_\bullet) \cong \mathrm{Tensor}_{E_*}(E_* M_\bullet)$

and $\pi_*(E_* Y_\bullet) \xrightarrow{\cong} E_* X$.

Crucial observation

$s\mathrm{Alg}_k$ = simplicial k-algebras = algebras in simplicial k-modules.

I.e. $A_\bullet \in s\mathrm{Mod}_k$ is equipped with $A_\bullet \otimes_k A_\bullet \longrightarrow A_\bullet$

Let $\mathcal{C}$ be any category with colimits and K a simplicial set. Then for $X_\bullet \in s\mathcal{C}$

$$K \otimes X \in s\mathcal{C} \text{ given by } (K \otimes X)_n = \coprod_{K_n} X_n .$$

Warning: If $X_\bullet \in s\underline{S}$ is a simplicial spectrum, then

$$K \otimes X_\bullet \not\simeq |K|_+ \wedge X_\bullet$$

Example: If $M_\bullet \in s\mathrm{Mod}_k$, then $K \otimes M_\bullet = k[K] \otimes_k M_\bullet$

If $X_\bullet \in s\underline{S}$ simplicial spectrum, then

$$[K \otimes X]_n = (K_n)_+ \wedge X_n$$

If $\mathcal{L}$ is any operad in simplicial sets and $\mathcal{C}$ has a symmetric monoidal $\wedge$ structure, then we have $\mathcal{L}$-algebras in $s\mathcal{C}$, i.e. $X_\bullet \in s\mathcal{C}$

$$\mathcal{L}(n) \otimes_{\Sigma_n} \underbrace{X_\bullet \wedge \dots \wedge X_\bullet}_{n} \longrightarrow X_\bullet$$

In particular, in each simplicial dimension k, $\mathcal{L}(n)_k$ is a set-operad and X_k is an $\mathcal{L}(n)_k$-algebra.

Example: $\mathcal{L}(n) = \mathrm{Ass}(n) = \Sigma_n$, as constant simplicial set
then $\mathcal{L}$-algebras in $s\mathrm{Mod}_k$ are precisely $s\mathrm{Alg}_k$.

Definition: An E_∞-operad $\mathcal{E}$ is a simplicial set operad such that
(1) for all n, k, $\mathcal{E}(n)_k$ is a free Σ_n-set and $\mathcal{E}(n)$ is weakly contractible.

(2) $\mathcal{E}(s\mathcal{C})$ = algebras over the operad $\mathcal{E}$ in $s\mathcal{C}$.

Proposition: 1) Let X be a simplicial spectrum and $\mathcal{E}$ an E_∞-operad in simplicial sets. Then if E_* has a Künneth spectral sequence, and $E_* X_n$ is projective for each n, then

$$E_*\big(\mathcal{E}(X_\bullet)\big) \cong \mathcal{E}(E_* X_\bullet)$$

Here $\mathcal{E}(-)$ = free algebra in $s\mathcal{C}$, with $\mathcal{C}$ understood

2) If $X_\bullet \in \mathcal{E}(s\underline{S})$, then the geometric realization $|X_\bullet|$ as a spectrum is an E_∞-ring spectrum.

Proposition: The category $\mathcal{E}(s\underline{S})$, $\mathcal{E}(s\mathrm{Mod}_k)$ are independent up to Quillen equivalence are independent of $\mathcal{E}$
(with $\pi_* E_*(-)$ equivalences resp. π_*-equivalences)

Proof of upper Prop, Part 1):

$$\big[E_*\, \mathcal{E}(X_\bullet)\big]_k = E_*\Big(\bigvee_{n\geq 0} (\mathcal{E}(n)_k)_+ \wedge_{\Sigma_n} X_k \wedge \dots \wedge X_k\Big)$$

↑ free Σ_n-set ?

$$\cong \bigoplus_{n \geq 0} E_*\big(\mathcal{E}(n)_k\big) \otimes_{E_*[\Sigma_n]} (E_* X_k)^{\otimes^n_{E_*}}$$

projectivity

$$= \mathcal{E}(E_* X_k)$$

Example: If $A_\bullet$ is a simplicial commutative k-algebra, then

$\mathcal{E} \longrightarrow \underline{\mathrm{Comm}}$ makes $A_\bullet$ into an object in $\mathcal{E}(s\mathrm{Mod}_k)$

In particular, this goes for the constant simplicial algebra.
If M is an A-module (for A discrete/constant), then

$$B_\bullet = A \ltimes K(M,n) \in \mathcal{E}(s\mathrm{Mod}_k).$$

E_∞- AQ cohomology:

$$H^n_{\mathcal{E}}(A/k, M) = \pi_0 \underbrace{\mathrm{map}_{\mathcal{E}(s\mathrm{Mod}_k)/A}\left(A, A \ltimes K(M,n)\right)}_{\mathcal{H}^n(A,M)}$$

derived mapping space.

Robinson-Whitehouse: Competing theory using Γ-homology.

Theorem: If A is a commutative algebra in E_*E-comodules (and certain hypotheses on E), there are successively defined obstructions

$$\theta_s \in H^{s+2}_{\mathcal{E}-E_*E}(A/k, A[s])$$

to realising A as an E_∞ ring spectrum.
There is a spectral sequence for analyzing the entire moduli space of E_∞-realizations of A.

Proposition: $H^*_{\mathcal{E}}(A/k; M)$ satisfies flat base change, transitivity, and vanishes when $k \to A$ is étale.

Flat base change:

$$\begin{array}{ccc} k & \xrightarrow{f} & L \\ {\scriptstyle g}\downarrow & & \\ A & & \end{array}$$

If f or g is flat, and M is an $A \otimes_k L$-module. Then

$$H^*_{\mathcal{E}}(A/k, M) \cong H^*_{\mathcal{E}}(L \otimes_k A/L; M)$$