In the case of $s\mathcal{S}^{Acs}$, $\pi_i^\natural(X, P)$ or $E_i^\natural(X)$ behave like htpy. grps. of a simple space - so it has Postnikov decomp.

$$\begin{array}{ccc} P_n X_\bullet & \longrightarrow & P_0 X_\bullet \\ \downarrow & & \downarrow \\ P_{n-1} X_\bullet & \longrightarrow & B(A, n+1) \end{array}$$

has $\pi_0^\natural$ and $\pi_{n+1}^\natural$

In the algebraic world, $E_* X_\bullet$ is a simpl. algebra in E_*E - comodules. Here we also has Postnikov sections in the simpl. direction and corresponding Eilenberg-MacLane objects $K(A, n+1)$. But

$$E_* B(A, n+1) \neq K(A, n+1)$$

defined using $\pi_*^\natural$

defined using $\pi_*(\pi_*(-, P))$.

The relationship is given by the spiral exact sequence. In particular,

$$P_{n+1}^{alg} E_* B(A, n+1) = K(A, n+1)$$

Prop The map of simpl. mapping spaces

$$\mathrm{Map}/_{P_0}(X, B(A,n+1)) \xrightarrow{E_*} \mathrm{Map}/_{E_*P_0}(E_*X, K(A,n+1))$$

is a weak equivalence.

pf Formal argument reduces to check the case $X = T(\Delta[q]/\partial\Delta[q] \wedge P)$ with $P \in \mathcal{P}$. This case is proved by calc.:

$$s\mathcal{S}^{A_\infty}(T(\Delta[q]/\partial\Delta[q] \wedge P), B(A,n+1))$$

$$= s\mathcal{S}(\Delta[q]/\partial\Delta[q] \wedge P, B(A,n+1))$$

$$= \pi_q^{\natural}(B(A,n+1), P).$$

//

Recall what we are doing.

E - some homology theory

X - multiplicative up to homotopy

Wish to find an A_∞-ring spectrum R and an E-homology equivalence

$$R \xrightarrow{\sim} X.$$

If E satisfies the Adams-Atiyah cond., we found a map, mult. up to htpy, from a free A_∞-ring spectrum

$$T \longrightarrow X \quad \text{s.t.}$$

$$E_*T \twoheadrightarrow E_*X$$

$$E_*T = T_{E_*}(P) \qquad P \text{ proj. } E_*\text{-mod.}$$

Find $T_1 \to T$ hitting the relations in $E_*(-)$ s.t. we have a push-out

$$\begin{array}{ccc} E_*T_1 & \longrightarrow & E_* \\ \downarrow & & \downarrow \\ E_*T & \longrightarrow & E_*X \end{array}$$

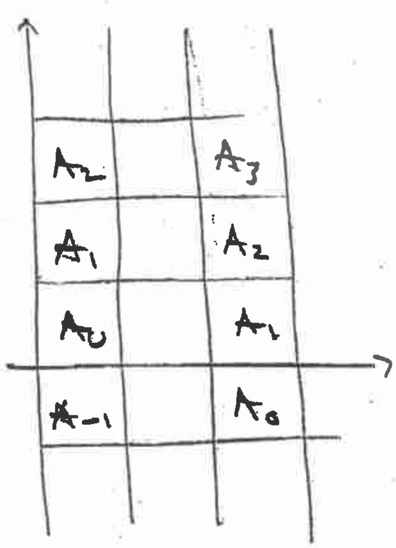
Make a simpl. A_∞-ring spectrum

$$X^{(1)} = T_1 \otimes_1 \Delta[1] \amalg_{T_1 \otimes \partial\Delta[1]} I_0 \amalg S^0 .$$

Then $E_* X^{(1)}$ is a simpl. assoc. algebra with $\pi_0 E_* X^{(1)} = E_* X$. To make something canonical, pass to $P_0 X^{(1)}$ (controls the $\pi_*^\natural$-groups) in the E-resolution model structure. Recall that if R is a simpl. A_∞-ring spectrum, then $\pi_*^\natural R$ are the D_1^2-groups of the spectral seq.

$$\pi_* E_* R \Rightarrow E_* |R| .$$

Let $A_* = E_* X$. Then $\pi_* E_* P_0 X^{(1)}$ is:



A_* ΩA_*

We wish to get rid of the ΩA_* part, i.e. to make a 1^{st} Postnikov approx. P_1 to X s.t.

$$\pi_n E_* P_1 = \begin{cases} E_* X & n=0 \\ 0 & n=1 \end{cases}$$

$$\pi_n^\natural P_1 = 0 \quad , n>1 ,$$

and, more generally, a n^{th} Postnikov approx. P_n to X s.t.

$$\pi_k E_* P_n = \begin{cases} E_* X & k=0 \\ 0 & 0<k\le n \end{cases}$$

$$\pi_k^\natural P_n = 0 \qquad k>n .$$

So the E^2 and D^2-groups should be

$$E^2: \quad \begin{array}{cccccccc} A_* & 0 & --- & 0 & 0 & \Omega^{n+1} A_* & 0 & - \\ 0 & & & & & n+2 & & \end{array}$$

$$D^2: \quad \begin{array}{cccccccc} A_* & \Omega A_* & -- & \Omega^n A_* & 0 & 0 & - & - \\ 0 & 1 & & n & & & & \end{array}$$

(It turns out that each P_n is contractible, but that the limit is X.)

$$\begin{array}{ccc} P_n & \longrightarrow & P_0 \\ \downarrow & & \downarrow \\ P_{n-1} & \longrightarrow & B(\pi_n^{\natural} P_n, n+1) = B(\Omega^n A_*, n+1) \end{array}$$

Having produced P_{n-1}, we need to produce the lower horizontal map. But this is captured entirely by algebra:

$$[P_{n-1}, B(\Omega^n A_*, n+1)] = [E_* P_{n-1}, K(\Omega^n A_*, n+1)].$$

Both $\pi_* E_* P_{n-1}$ and $\pi_* K(\Omega^n A_*, n+1)$ are

$$\underset{0}{A_*} \circ \cdots \circ \underset{n+1}{\Omega^n A_*} \circ \cdots$$

and we are looking for maps from $E_* P_{n-1}$ to $K(\Omega^n A_*, n+1)$, in the algebraic model cat, which are isomorphisms.

$$M_* \longrightarrow E_* P_{n-1} \longrightarrow A_*$$

simpl. algebra

with

$$\pi_k M_* = \begin{cases} \Omega^n A_* & k = n+1 \\ 0 & \text{else} \end{cases}$$

so $E_* P_{n-1}$ looks like a square-zero extension of A_* by $\Omega^n A_*$ in degree $n+1$. Similarly,

$$M_* \to K(\Omega^n A, n+1) \twoheadrightarrow A_*$$

but this extension splits. So the obstruction for an too

$$E_* P_{n-1} \xrightarrow{k} K(\Omega^n A, n+1)$$

to exists, is an element of

$$\mathrm{Der}^{n+2}_{\mathrm{Cat}}(A_*, \Omega^n A_*)$$

where Cat is e.g. assoc. E_*-alg., assoc. algebras in E_*E-comodules, etc.

Describe this obstruction in more detail: We form the following push-out in the cat. of simpl. algebras

$$\begin{array}{ccc} M_* & & \\ & \searrow & \\ & E_* P_{n-1} & \longrightarrow & A_* \\ & \downarrow & \text{p.o.} & \downarrow \\ & A_* & \longrightarrow & K' \end{array}$$

then kill all $\pi_i K'$ for $i > n+2$ to get a htpy. cartesian sq.

$$\begin{tikzcd} E_* P_{n-1} \arrow[r] \arrow[d] & A_* \arrow[d] \\ A_* \arrow[r] \arrow[ur, dashed] & K(\Omega^n A_*, n+2) \end{tikzcd}$$

Then a splitting of the left-hand map exists if and only if the lower horizontal map factors through a dotted map as indicated. The lower horizontal map defines an element of

$$\mathrm{Der}^{n+2}_{\mathrm{Cat}}(A_*, \Omega^n A_*)$$

and this is the obstruction. So to summarize, the obstr. to forming P_n from

P_{n-1} is an element of

$$\mathrm{Der}^{n+2}_{\mathrm{Cat}}(A_*, \Omega^n A_*).$$

The obstr. to P_n' and P_n'' being equiv. is an element of

$$\mathrm{Der}^{n+1}_{\mathrm{Cat}}(A_*, \Omega^n A_*).$$

Let

$$\mathcal{M}_n = \text{moduli space of } n\text{-Postnikov approx. to } X.$$

Then there is a htpy cartesian sq.

$$\begin{tikzcd} \mathcal{M}_n \arrow[r] \arrow[d] & \mathcal{M}_0 \arrow[d] \\ \mathcal{M}_{n-1} \arrow[r] & \check{H}^{n+2}(A, \Omega^n A) \end{tikzcd}$$

where $A = E_* X$ and

$$\mathcal{M}_0 = B\,\mathrm{aut}(A) = B\,\mathrm{haut}(P_0)$$

$$\check{H}^{n+2}(A, \Omega^n A) = \underset{\text{maps}}{\text{simp. alg.}}(A, K(\Omega^n A, n+2)) \times_{\mathrm{aut}(A)} E\,\mathrm{aut}(A).$$

Suppose Y = n^{th} approx. to X:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & P_0 Y \arrow[d] \\ P_{n-1} Y \arrow[r] & B(\Omega^n A, n+1) \end{tikzcd}$$

then

$$m_n = m\,(P_{n-1} Y \to B(\Omega^n A, n+1) \leftarrow P_0 Y).$$

The square on the previous page is derived from this.

21 Nov. TMF

E - Landweber exact

$$\left.\begin{array}{l} E_{odd} = 0 \\ \pi_2 E \otimes_{\pi_0 E} \pi_{-2} E \xrightarrow{\approx} \pi_0 E \end{array}\right\} \begin{array}{l} \text{"even" condition} \\ \text{"periodic" "} \end{array}$$

$\Rightarrow E^0(\mathbb{C}P^\infty) =$ ring of functions on some formal gp G_E.

Landweber's criterion:

the map G_E is flat iff $(p, v_1, v_2, \dots)$ is regular.

$$\begin{array}{c} \operatorname{Spec} \pi_0 E \\ \downarrow G_E \\ \mathcal{M}_{FG} \end{array}$$

Suppose we have

F another even periodic cohomology theory

$E \wedge F$ two formal gps

- one from E
- one from F

$(E\wedge F)^0(\mathbb{C}P^\infty)$ iso

$$\begin{tikzcd} \operatorname{Spec} \pi_0 E\wedge F \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr] & & \\ & X \times_{\mathcal{M}_{FG}} Y \arrow[r] \arrow[d] & \operatorname{Spec} \pi_0 E = Y \arrow[d, "G_E"] \\ & X = \operatorname{Spec} \pi_0 F \arrow[r, "G_F"'] & \mathcal{M}_{FG} \end{tikzcd}$$

iso if one of G_E, G_F is flat

(Pf: another time)

Ex.

take E_n

this is where $\pi_0 E_n$ = ring of functions on universal deformations of a FG of height n.

e.g. $\pi_0 E_n = W[[u_1, \dots, u_{n-1}]]$

where W = ring of Witt vectors of $\overline{\mathbb{F}}_p$

- satisfies Landweber's criterion $\Rightarrow$ E_n is a multiplicative homology theory.

Q: Can we make E_n a A_∞ (or better) ring?

Consider $R \to E_n$.

We want to say only alg. things about R

It's an equivalence in homology, i.e.

$$(E_n)_* R \longrightarrow (E_n)_* E_n$$

We need more informations, for example

$$(E_n)_* \mathbb{C}P^\infty = \bigoplus^\infty E_{n*}$$

$$(E_*)\Big(\bigvee_{n=0}^{\infty} S^{2n}\Big) = \bigoplus E_{n*}$$

more informations to distinguish between the spectra.

X $\quad (E_n)_* = \pi_*(E_n \wedge X)$

$X \to E_n \wedge X$ $\quad$ must look at $E_n \wedge E_n \wedge X$ $\quad$ so $\pi_*(E_n \wedge E_n \wedge X)$

Introduce this: $E_{n*}E_n = \pi_* E_n \wedge E_n$ $\quad$ where E = ordinary mod p homology

E_*E = dual Steenrod algebra

Recall:

$$\begin{tikzcd}
(E_n)_0 E_n \arrow[d, equal] & \\
\operatorname{Spec} \pi_0 E_n \wedge E_n \arrow[r] \arrow[d, "\boxed{(**)\ \text{flat}}"'] & \operatorname{Spec} \pi_0 E_n \arrow[d, "\text{flat}"] \\
\operatorname{Spec} \pi_0 E_n \arrow[r] & \mathcal{M}_{FG}
\end{tikzcd}$$

$$(E_n \wedge E_n)_* X \overset{(**)}{\Downarrow} = \pi_*(E_n \wedge E_n) \otimes_{\pi_0 E_n} E_{n*} X$$

$$= \pi_0(E_n \wedge E_n) \otimes_{\pi_0 E_n} E_{n*} X$$

$$\pi_* E_n \wedge E_n \wedge E_n =$$

$$= E_{n*} E_n \otimes_{E_{n*}} E_{n*} E_n$$

for $n = 1$

E = p-adic K-theory

$$\pi_0(E_n \wedge E_n)^\wedge_p = \mathcal{C}_{cts}(\mathbb{Z}_p^*, \mathbb{Z}_p)$$

$$\pi_* E_n \wedge E_n \wedge E_n = E_{n*} E_n \otimes_{E_{n*}} E_{n*} E_n$$

$$\uparrow \uparrow \uparrow$$

$$\pi_* E_n \wedge E_n = E_{n*} E_n$$

maps given by

$$x \otimes 1 \quad \psi(x) \quad 1 \otimes x$$

$$\nwarrow \uparrow \nearrow$$

$$x$$

ψ = a co-multiplication

and it makes $E_{n*} E_n$ a co-algebra over E_{n*}

$$(E_n)_* X = \pi_* E_n \wedge X$$
$$\downarrow \downarrow$$
$$\pi_*(E_n \wedge E_n \wedge X)$$
$$\|$$
$$(E_{n*} E_n) \otimes_{E_{n*}} E_{n*} X$$

$$a \mapsto 1 \otimes a$$

makes $E_{n*}X$ an E_*E co-module

$E_{n*} R \simeq E_{n*} E_n$

as an algebra in $E_{n*}E_n$ co-modules

it's an associative algebra w/ an extra structure

this is the algebraic thing we want.

A = a ring , flat/$\mathbb{Z}$
M = an abelian gp
N = an A-module

$$\begin{array}{ccc} A \otimes M & \xrightarrow{A\text{-mod}} & N \\ 1\otimes m \;\uparrow & \Updownarrow & \\ m \quad M & \xrightarrow[\text{abel gp}]{} & N \end{array}$$

co-module analogue:

$(E_n)_* E_n$ = co-algebra
M = E_*-module
N = E_*E co-module

want a map $N \xrightarrow{\text{co-mod}} E_*E \otimes_{E_*} M$

it's the same as

$$N \xrightarrow[E_*\text{-mod}]{} M$$

By last lecture:

the obstructions to existence of an A_∞ ring R
for which $E_{n*}R \approx E_{n*}E_n$ as $E_{n*}E_n$-comodules
are in

$$\mathrm{Der}^{n+1}_{\substack{E_{n*}E_n\\ \text{co-mod}}}(\underbrace{E_{n*}E_n}_{A}, \underbrace{\Omega^n E_{n*}E_n}_{M})$$ ← previous notation

and the obstructions to uniqueness

$$\mathrm{Der}^{n}_{E_{n*}E_n}(E_{n*}E_n, \Omega^n E_{n*}E_n).$$

(These groups are actually:)
the existence ~~obstructions~~ are in

$$\mathrm{Der}^{n+1}_{E_{n*}}(E_{n*}E_n, \Omega^n E_{n*})$$

and the uniqueness ones

$$\mathrm{Der}^{n}_{E_{n*}}(E_{n*}E_n, \Omega^n E_{n*}).$$

Claim: $\mathrm{Der}^s_{\underbrace{E_{n*}}_{A}}(\underbrace{E_{n*}E_n}_{\Gamma}; \underbrace{\Omega^n E_{n*}}_{B}) = 0 \quad \forall s$

Pf:
Since E_{n*} is p-complete, it suffices to show

(*) $$\mathrm{Der}^s_{(E_{n*})\otimes \mathbb{Z}/p}((E_n)_*E_n \otimes \mathbb{Z}/p, (E_{n*})\otimes \mathbb{Z}/p) = 0$$

Because want $\mathrm{Der}_A(\Gamma, B)$
suffices $\mathrm{Der}_A(\Gamma, B/p^n) \quad \forall n$

use Milnor sequence (inverse lim of coh. of cx)

$$\varprojlim_n{}^1 \mathrm{Der}^{s+1}(\Gamma, B/p^n) \to \mathrm{Der}_A^s(\Gamma, B) \to \varprojlim_n \mathrm{Der}_A(\Gamma, B/p^n)$$

use LES and induction

$$p^{n-1}(B/p) \longrightarrow B/p^n \longrightarrow B/p^{n-1}$$

(*) will be a consequence of:

relative (Frobenius)^n $(E_n)_* E_n/p \circlearrowleft_{E_{n*}/p}$ is iso