16.10.03

Today we'll construct $L_{K(2)} \mathcal{O}^{top}$ (completed at some prime p)

At $p=2$, the candidate was

$$L_{K(2)} tmf = L_{K(2)} \mathcal{O}^{top}(\mathcal{M}_{ell}) \simeq KO[\alpha]$$

This is supposed to be E_∞, so it has ψ and ~~all~~ θ-operations.

Recall the ring of modular forms

$$\bigoplus_n H^0(\mathcal{M}_{weier}, \omega^n) = \mathbb{Z}[c_4, c_6, \Delta] \big/ c_4^3 - c_6^2 = 1728 \cdot \Delta$$

The ring of rational functions on $\mathcal{M}_{weier}$ is $\mathbb{Z}(j)$ with $j = \frac{c_4^3}{\Delta}$ of weight 0. The above α is a modular function, and in fact

$$\alpha = \frac{1}{j} \mod 2.$$

In terms of functions on the upper half plane, we have q-expansions

$$c_4 = 1 + 240 \sum \sigma_3(n) q^n \qquad \Big(\sigma_k(n) = \sum_{d \mid n} d^k\Big)$$

$$\Delta = q \cdot \prod (1-q^n)^{24} = \sum \tau(n) q^n$$

($\Rightarrow$ can get q-expansion for j).

In terms of q-expansions, the operation ψ is given by $(\psi f)(q) = f(q^2)$ at $p=2$.

We'll construct $L_{K(2)} tmf$ by "generators and relations" in $K(2)$-local E_∞-ring spectra.

<u>Prop.</u> If X is $K(1)$-local, then the sequence

$$X \longrightarrow KO \wedge X \xrightarrow{\psi^3 - Id} KO \wedge X$$

is a fibration

(beware: the smash products are in the $K(1)$-local category, so they are 2-adically completed!)

Example: $X = L_{K(1)} S^0$; look at homotopy in low dimensions

$$\begin{tikzcd}
\pi_0 KO \arrow[r, "\psi_3 - 1"] \arrow[d, equal] & \pi_0 KO \arrow[r] \arrow[d, equal] & \pi_{-1} L_{K(1)} S^0 \arrow[r] \arrow[d, equal] & \pi_{-1} KO \arrow[d, equal] \\
\mathbb{Z}_2 \arrow[r, "0"'] & \mathbb{Z}_2 \arrow[r, "\cong"'] & \mathbb{Z}_2 \arrow[r] & 0
\end{tikzcd}$$

$$1 \longmapsto \zeta$$

From now on, everything is $K(1)$-local / localized, so "S^0" stands for "$L_{K(1)} S^0$", etc.

$$\zeta : S^{-1} \longrightarrow S^0 \qquad (\in \pi_{-1} L_{K(1)} S^0)$$

Since $\pi_{-1}(KO[[x]]) = 0$, we have to get rid of ζ.

Set $M_\zeta = \text{cofibre}\,(S^{-1} \xrightarrow{\zeta} S^0)$. Then

$$KO_0(M_\zeta) = \underset{a}{\mathbb{Z}_2} \oplus \underset{b}{\mathbb{Z}_2} \qquad \text{with } \psi_3(a) = a, \quad \psi_3(b) = b + a.$$

Suppose R is such that $\pi_0 R \longrightarrow \pi_0(KO \wedge R)$ is injective.
Then $[M_\zeta, R] = \{a, b \in KO_0 R \mid \psi_3(a) = a,\ \psi_3(b) = b + a\}$.

Now build universal E_∞ $K(1)$-local ring spectrum:

$$\begin{tikzcd}
\mathrm{Sym}(S^{-1}) \arrow[r, "0"] \arrow[d, "\zeta"'] & S^0 \arrow[d] \\
S^0 \arrow[r] & T_\zeta
\end{tikzcd}$$

pushout in E_∞-$K(1)$-local rings.

If R is E_∞ and $\pi_0 R \hookrightarrow KO_0 R$, then

$$\pi_0 E_\infty(T_\zeta, R) = \{x \in KO_0 R \mid \psi_3(x) = x + 1\}$$

(the above "a" has to be the unit now).

(Lots of this is in an Appendix to a paper of Gerd Laures, see gerd.laures.com

$$\begin{tikzcd} KO \wedge \mathrm{Sym}(S^{-?}) \arrow[r, "0"] \arrow[d, "0 = KO\wedge\xi"'] & KO \arrow[d] \\ KO \arrow[r] & KO \wedge T_\xi \end{tikzcd} \quad = \mathrm{Sym}_{KO}(S^0) \simeq KO \wedge \mathrm{Sym}(S^0)$$

McClure: $\pi_0(KO \wedge T_\xi) = \mathbb{Z}_2[x, \theta x, \theta^2 x, \ldots]$, $\psi_3(x) = x+1$.

We use the fibration from earlier to get at the homotopy of T_ξ.
Set $b = \psi(x) - x \in KO_0 T_\xi$. Since ψ and ψ_3 commute, we get

$$\psi_3(b) = b \quad \text{and (with some work)}$$

$$\pi_* T_\xi = KO_*[b, \theta b, \theta^2 b, \ldots]$$

Warmup calculation: Construct KO (= presentation of KO, using KO)

$$\pi_0(KO \wedge KO) = \mathrm{Cont}(\mathbb{Z}_2^\times/\{\pm 1\}, \mathbb{Z}_2)$$

The number "3" above is really a generator of $\mathbb{Z}_2^\times/\{\pm1\}$, so

$$\cong \mathrm{Cont}(\mathbb{Z}_2, \mathbb{Z}_2)$$

$3 \in \mathbb{Z}_2^\times/_{\pm}$
$\updownarrow$
$1 \in \mathbb{Z}_2$

$\mathbb{Z}_2 \ni a = \sum \alpha_i \cdot 2^i$ with $\alpha_i \in \{0,1\}$, then we can view α_i = i-th Witt component as a function

$$\alpha_i : \mathbb{Z}_2 \longrightarrow \{0,1\}.$$

Then

$$C(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[\alpha_0, \alpha_1, \ldots] / \alpha_i^2 = \alpha_i$$

$$\parallel$$

$$\varprojlim_{\ell} \varinjlim_{k} C(\mathbb{Z}/2^k, \mathbb{Z}/2^\ell)$$

This is a θ-algebra with

$$\psi(f) = f \quad \text{and} \quad \theta(f) = \frac{f^2 - f}{2}$$

$$(\psi_3(f))(t) = f(t+1).$$

Get map of θ-algebras

$$\begin{array}{ccc} KO_0 T_\zeta = \mathbb{Z}_2[x, \theta x, \theta^2 x, \ldots] & & \ni x \\ \downarrow & & \downarrow \\ KO_0 KO = \mathbb{Z}_2[\alpha_i]/\alpha_i^2 = \alpha_i & & f : S \mapsto S \end{array}$$

θ-algebra morphism compatible with ψ^3.

One checks that $\theta^i x \mapsto \alpha_i \mod 2, \ g(\alpha_0, \ldots, \alpha_{i-1})$

The element $b = \psi x - x$ is in the kernel of $KO_0 T_\zeta \to KO_0 KO$.
Since b comes from $\pi_0 T_\zeta$, we can form the E_∞-pushout

$$\begin{tikzcd} Sym(S^0) \arrow[r, "0"] \arrow[d, "b"'] & S^0 \arrow[d] \\ T_\zeta \arrow[r] & R \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Apply KO_0:

$$\begin{array}{ccc} y & & \mathbb{Z}[y, \theta y, \theta^2 y, \ldots] \longrightarrow \mathbb{Z} \\ \downarrow & & \downarrow \\ b = \psi x - x & & \mathbb{Z}[x, \theta x, \theta^2 x, \ldots] \end{array}$$

Check $\theta^n y \mapsto (\theta^n(x)^2 - \theta^n(x)) +$ lower terms mod (check formulas ...)

$\Rightarrow \ KO_0 R = C(\mathbb{Z}_2, \mathbb{Z}_2)$, and thus $R \simeq KO$.

<u>Conclusion:</u> in $K(1)$-local E_∞ ring spectra, has cell decomposition

$$KO \simeq * \cup_\zeta e^0 \cup_b e^1, \text{ thus}$$

$$E_\infty(KO, R) \Leftarrow H^*_{cell}(KO, \pi_* R)$$

a kind of AQ-cohomology from Paul Goerss' lectures.

~~Back~~ to making tmf as $KO[[x]]$ with "funny multiplication".

Suppose R is a $K(2)$-local elliptic spectrum (tmf is supposed to be the "universal" one ...) For

$$T_\zeta \longrightarrow R$$ we need a "universal class" $x \in KO_0 R$ with $\psi_3(x) = x+1$

Consider $c_4/v_1^4 \in \pi_0(KO \wedge R)$ where $c_4 \in \pi_8 R$ and $v_1^4 \in \pi_8 KO$

$$\psi_3(c_4) = c_4$$
$$\psi_3(v_1^4) = 3^4 v_1^4$$

Set $x_{mod} = \dfrac{\log(c_4/v_1^4)}{\log(3^4)}$, then $\psi_3(x_{mod}) = x_{mod}+1$.

Recall $b = f(x) - x \in \pi_0 T_\zeta$

Set $b_{mod} = f(x_{mod}) - x_{mod}$, a modular function:

$$c_4 = 1 + 240 \cdot \sum_n \sigma_3(n)\, q^n$$

$$b_{mod} = \sum \sigma_3^*(n) \cdot q^n \mod 8, \text{ where } \sigma_3^*(n) = \sum_{\substack{d|n \\ (d,p)=1}} d^3$$

Famous congruence of Ramanujan:

$$\tau(n) \equiv \sigma_3^*(n) \mod 8$$

Recall $(\alpha =)$ $\dfrac{1}{j} = \dfrac{\Delta}{c_4^3} \equiv \sum \tau(n)\, q^n \mod 8$

$$\equiv b_{mod} \mod 8$$

This implies $g(b_{mod}) = h(b_{mod})$ for some power series h

Since $1/j$, hence b_{mod} is a uniformizer for the ring of modular functions.

Attach another cell:

$$\begin{tikzcd}
\mathrm{Sym}(S^0) \arrow[r, "0"] \arrow[d] & S^0 \arrow[d] \\
T_\xi \arrow[r, "\Gamma"] & R_1
\end{tikzcd}$$

$\vartheta(b)=h(b)$

hold for modular forms, but not in T_ξ

One can check now that $R_1 = KO[\![x]\!]$, so $R_1 = L_{K(1)}\,tmf$.

This R_1 has "correct" homotopy groups and ϑ-algebra structure, but needs more justification to call it $L_{K(1)}\,tmf$.

Summary:

$$L_{K(1)}\,tmf \simeq \ast \cup_\xi e^0 \cup_{\vartheta(b)-h(b)} e^1$$

<u>17.10.03</u>

Spectra + Stable homotopy

"too many maps"

Idea: use global objects in algebraic geometry, like elliptic curves, to regulate the homotopy theory.

$$\begin{tikzcd}
\mathcal{E} \arrow[d, "\simeq"'] \arrow[r, "\text{defined by some condition: } E_\infty\text{-structures}"] & \text{Stable homotopy (Landwebr spectra)} \arrow[d, squiggly] \\
\text{(Elliptic curves)} \arrow[ur, dashed] \arrow[r] & \text{Algebra (Formal groups)}
\end{tikzcd}$$

enough objects by obstruction theory — $\mathcal{E}$

Stacks

Not all elliptic curves come from topology, only "good" families.
We need to understand not just properties of the curve C, but of the family $C \longrightarrow \operatorname{Spec} S$. If there was a classifying "space", the property can be expressed in terms of the classifying map

$$\operatorname{Spec} S \longrightarrow \mathcal{M}_{Weir} / \mathcal{M}_{ell}$$

classifies the curve C

"good" = (formally) étale maps

tmf = cohomology theory corresponding to $\mathcal{M}_{Weier}$

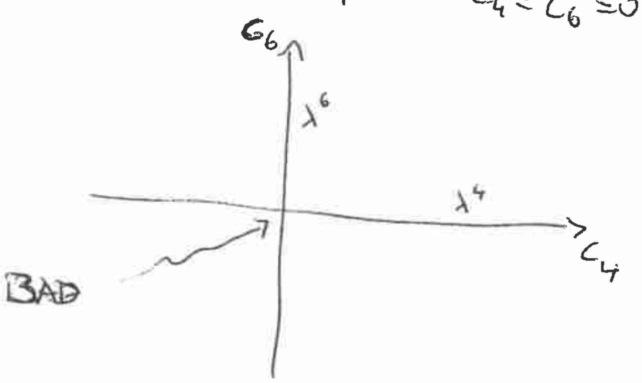
$$H^s(\mathcal{M}_{Weier}; \omega^n) \Rightarrow \pi_{2n-s}\, tmf$$

$$H^0(\mathcal{M}_{Weir}, \omega^n) = \text{modular forms of wt } n.$$

Invert 6:

Then $$\bigoplus_n H^s(\mathcal{M}_{weir}, \omega^n) = \begin{cases} \mathbb{Z}[\tfrac{1}{6}][c_4, c_6] & \text{for } s=0 \\ 0 & \text{for } s>0 \end{cases}$$

$\mathcal{M}_{Weir}$ has the bad point $c_4 = c_6 = 0$



$\mathcal{M}_{ell} = \mathcal{M}_{Weier} \setminus \{c_4 = c_6 = 0\}$ still has one singular (generalised) elliptic curve.

We removed the bad point to make the height of formal groups ≤ 2 and get Landweber exactness.

Set $U_0 = c_4^{-1} \mathcal{M}_{ell}$, $U_1 = c_6^{-1} \mathcal{M}_{ell}$

Mayer - Vietoris

$$H^*(U_0) \oplus H^*(U_1) \longrightarrow H^*(U_0 \cap U_1) \longrightarrow H^*(\mathcal{M}_{ell})$$

$$\begin{array}{ccccc} H^0(\mathcal{M}_{ell}) & \longrightarrow & c_4^{-1}\mathbb{Z}[\frac{1}{6}, c_4, c_6] & & \\ \shortparallel & & \oplus & \longrightarrow & (c_4 c_6)^{-1} \mathbb{Z}[\frac{1}{6}, c_4, c_6] \\ \mathbb{Z}[\frac{1}{6}, c_4, c_6] & & c_6^{-1} \mathbb{Z}[\frac{1}{6}, c_4, c_6] & & \end{array}$$

$\frac{1}{c_4 c_6} \in H^1(\omega^{-10}) = \mathbb{Z}[\frac{1}{6}]$

$H^1(\omega^k) =$ dual to $H^0(\omega^{-10-k})$

$\pi_* \mathcal{O}^{top}$:

-29	-21		0	8	12	20
$\frac{1}{c_4^2 c_6}$	$\frac{1}{c_4 c_6}$	$\uparrow$ "long gap"	1	c_4	c_6	$c_4 c_6$

$$(\pi_* \mathcal{O}^{top})(\mathcal{M}_{ell})[0, \infty) =: tmf$$

(removing bad point only introduces negative homotopy groups !)

$H^*(\mathcal{O}^{top}(\mathcal{M}_{ell}); \mathbb{Z}/p) = 0$, whereas $H^*(tmf, \mathbb{Z}/p)$ is interesting.

e.g. $H^*(tmf, \mathbb{Z}/2) = \mathcal{A}_2 \otimes_{A_2} \mathbb{Z}/2$

A_2 = subalgebra of $\mathcal{A}_2$ generated by Sq^1, Sq^2, Sq^4.

Can compute $\pi_* tmf$ via modular forms, but also differently via Adams spectral sequence (Rem: this has "forced upon you" differentials !)

We will construct sheaf of spectra $\mathcal{O}^{top}$ on $\mathcal{M}_{Ell}$.

At a prime p: Hasse square

$$\begin{tikzcd} (\mathcal{O}^{top})^{\wedge}_{p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{K(1)} \mathcal{O}^{top} \arrow[d] \\ L_{K(2)} \mathcal{O}^{top} \arrow[r] & L_{K(1)} L_{K(2)} \mathcal{O}^{top} \end{tikzcd}$$

Do it on enough affine families, to get on a basis in the étale topology. Then sheafify.

We need the values of $\mathcal{O}^{top}$ on

$$\{ \operatorname{Spec} R \xrightarrow{\text{étale}} \mathcal{M}_{Ell} \}$$

(Rk: there is a slightly different kind of obstruction theory for getting $L_{K(2)} \mathcal{O}^{top}$ ($=$ Γ-cohomology, AQ^{*}, no Dyer-Lashof operations)

$L_{K(1)} \mathcal{O}^{top}$ need free E_∞-ring spectra. ("embrace Dyer-Lashof")

That's why we don't know how to build $\mathcal{O}^{top}_{p}$ straight away.)

$L_{K(2)} \mathcal{O}^{top}$ — super singular elliptic curves

Serre-Tate deformation theory: sub-story of height 2 formal grps

$\updownarrow$ Haynes

$K(2)$-local E_∞ ring spectra

$L_{K(1)} \mathcal{O}^{top}$: ① produce a candidate for $L_{K(1)}$ tmf

②
$$\left(\begin{array}{c} \text{Etale } E_\infty \text{ elliptic spectra} \\ \text{over } R \end{array} \right) \longrightarrow \left(\begin{array}{c} \operatorname{Spec} \pi_0 E \\ \downarrow \text{étale} \\ \mathcal{M}_{ell}^{ord} \end{array} \right)$$

Show: is an equivalence

Here is a definition that works: $R \to E$ map of E_∞-ring spectra is *étale* if $K_* R \to K_* E$ is étale.

Here $K_* R := \pi_* \left(\operatorname{holim}_{\leftarrow} \left(K \wedge R \wedge M(p^n) \right) \right)$ completed !

$$
\begin{tikzcd}
\operatorname{Spec}(\pi_0 K \wedge E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec} \pi_0 E \arrow[d] \arrow[dd, bend left=60] \\
\operatorname{Spec}(K_0 R) \overset{!}{=} \operatorname{Isos}(\hat{C}, \mathbb{G}_m) \arrow[r, "\text{Galois !}", "\text{covering ?}"'] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}^{ord}_{Ell} \arrow[d] \\
\operatorname{Spec} \pi_0 K = \mathbb{Z}_p \arrow[r, "\hat{\mathbb{G}}_m"', "\text{étale covering}"'] & \mathcal{M}^{ht=1}_{FG}
\end{tikzcd}
$$

flat = Landweber exact

Need to show that $\operatorname{Spec}(K_0 R) = \operatorname{Iso}(\hat{C}, \mathbb{G}_m)$

Since "étale" is local, suffices to check: flat, [illegible] (...)

Assume this point for now, $\operatorname{Spec} K_0 R = \operatorname{Iso}(\hat{C}, \mathbb{G}_m)$

Homotopy type of moduli space of E is determined by a standard obstruction theory

$$H^*_{AQ,\, K_*K\text{-comod},\, \text{Galois grp}}(K_0 E / K_0 R) = 0$$

Since the map $K_0 R \to K_0 E$ is étale.

$\Rightarrow$ there is a unique such E and all mapping spaces are purely algebraic.

Other point: produce enough étale maps to cover $\mathcal{M}_{ell}$!

$\Rightarrow$ $\mathcal{M}_{ell}$ has étale cover by affines.

Mike Hopkins
17.10.03
Questions / Outlook

Pts: Weierstraß cubic + coordinate on its formal group

$$\begin{tikzcd}
\operatorname{Spec} \pi_0(MP \wedge tmf) \arrow[r] \arrow[d] & \mathcal{M}_{Ell} \arrow[d] \\
\operatorname{Spec} L \arrow[r] & \mathcal{M}_{FG}
\end{tikzcd}$$

← Stack pullback

C Weierstraß cubic

$\downarrow$

$\hat{C}$ formal grp

$$\operatorname{Spec} L = \pi_0 MP = \pi_* MU$$

formal grp + coordinate

Coordinate: local parameter near e

Lemma: Suppose C = Weierstraß cubic, t: local parameter near ∞
$\Rightarrow$ There is a unique equation for C

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

such that $$\frac{x}{y} = t + O(t^5)$$

Pf: Since unique, the question is local, so can assume some Weierstraß equation.

Scaling: $\left.\begin{array}{l} x \mapsto \lambda^2 x \\ y \mapsto \lambda^3 y \end{array}\right\} \frac{x}{y} \mapsto \lambda \cdot \frac{x}{y}$ use up choice of λ.

Still have: $\left.\begin{array}{l} x \mapsto x + r \\ y \mapsto y + sx + t \end{array}\right\}$ 3 degrees of freedom to fix up t^2, t^3, t^4.

Given $C^1 +$ z local parameter, get unique Weierstraß equation.

Then $$w = \frac{x}{y} = z + \sum_{i \geq 5} \alpha_i z^i$$

$$\Rightarrow \quad MV_{\delta}\, tmf = \mathbb{Z}[a_1, \dots, a_6][\alpha_5, \alpha_6, \alpha_7, \dots]$$

__Outlook__

$$\begin{pmatrix}\text{framed} \\ \text{manifolds}\end{pmatrix} \xrightarrow{\text{Pontryagin}} \begin{pmatrix}\text{stable homotopy} \\ \text{grps}\end{pmatrix}$$

$$(BO\langle 8\rangle/\text{String structures}) \longrightarrow tmf \; < \begin{matrix}\text{modular forms} \\ K(1)\ \&\ K(2)\ \text{local homotopy}\end{matrix}$$

$$(\text{Spin structures}) \longrightarrow (K\text{-local homotopy})$$

__Questions:__ What is the geometric interpretation of tmf?

__Question (Stolz)__: What is the geometric interpretation of string structures?

Stolz-Teichner ~~once~~ gave an answer to this question.
They also produce a cohomology theory which is geometrically defined and a candidate for tmf.

Kriz-Hu have produced another candidate.

Kervaire-Milnor exotic spheres

$$\Theta_k = \begin{array}{l}\text{Group of } h\text{-cobordism classes}\\ \text{of smooth structures on } S^{4k-1}\end{array} \cong \mathbb{Z}/d_k \times \text{subquotient of } \pi_* S^0$$

where $d_k = 2^{?} \times (2^{?} - 1) \cdot \text{numerator}\left(\frac{B_{2k}}{4k}\right)$

Hirsch-Mazur : $\Theta_k \cong \pi_{4k-1} PL/O$

Is there a map

$$PL/O \longrightarrow R \quad \text{with} \quad \pi_{4k-1} R = \mathbb{Z}/d_k \quad ?$$

Fibration of Ω^∞-loop spaces ⇝ spectra

$$\begin{tikzcd}
PL/String \arrow[d] \\
G/String \arrow[d] \\
G/PL
\end{tikzcd}
\qquad
\begin{tikzcd}
pl/string \arrow[r] \arrow[d] & tmf(2)^{\times}_{cusp} \arrow[d] \\
gl_1 S^0/string \arrow[r, "\text{Ochanine}"] \arrow[d] \arrow[dr, bend left, "\text{signature}"] & tmf(2)^{\times} \arrow[d, "cusp"] \\
gl_1 S^0/pl \arrow[r, "\simeq", "\text{Sullivan}"'] & KO^{\times}
\end{tikzcd}$$

$G = Gl_1 S^0$ $\quad gl_1$

$$\begin{tikzcd}
 & R \arrow[dr] & & \\
pl/string \arrow[ur, dashed] \arrow[rr] & & tm(2)^{\times}_{cusp} \arrow[r, "\log"] \arrow[d] & tmf(2)_{cusp} \arrow[d] \\
 & & tmf(2)^{\times} \arrow[r, "\log"] & tmf(2)
\end{tikzcd}$$

fibration

$\pi_* R$ are described by the zeroes of a p-adic L-function of Mazur-Wiles.

Ravenel: torsion in $\pi_* \mathrm{tmf}$?

$\pi_3 \mathrm{tmf} = \mathbb{Z}/24$ built in the SS from an extension

$$H^1(\omega^2) = \mathbb{Z}/12$$
$$H^3(\omega^3) = \mathbb{Z}/2$$

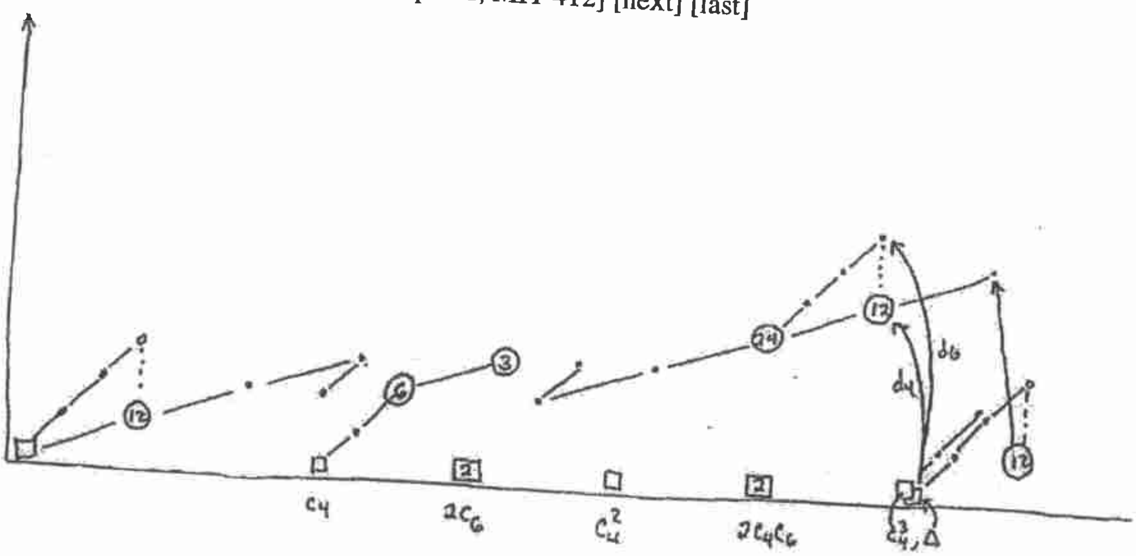
In Weierstraß equation $y^2 + \ldots = x^3 + \ldots$, x has a double pole at ∞. Is there an invariant function with at most double pole at ? Riemann-Roch: the space of such is 2-dimensional, containing the constant functions

$$\underline{\mathbb{1}} \subseteq \begin{matrix}\text{functions with} \\ \text{at most double pole}\end{matrix} \twoheadrightarrow \omega^2$$

A choice of x is a splitting. But the sequence defines a class in $\mathrm{Ext}^1(\omega^2, \mathbb{1}) = H^1(\mathcal{M}_{ell}, \omega^2)$.

There is a good choice for "$12 \cdot x$", so the class in Ext^1 has order 12

$SU(2)$ as framed manifold is detected in $\pi_3 \mathrm{tmf}$ coming from $H^2(\omega^5)$.



E_4-term on, of the spectral sequence

$$H^s(\mathcal{M}_{ell}; \omega^{\otimes t}) \Rightarrow tmf_{2t-s}$$

$\bullet = \mathbb{Z}/2$

$(n) = \mathbb{Z}/n$

$\square = \mathbb{Z}$