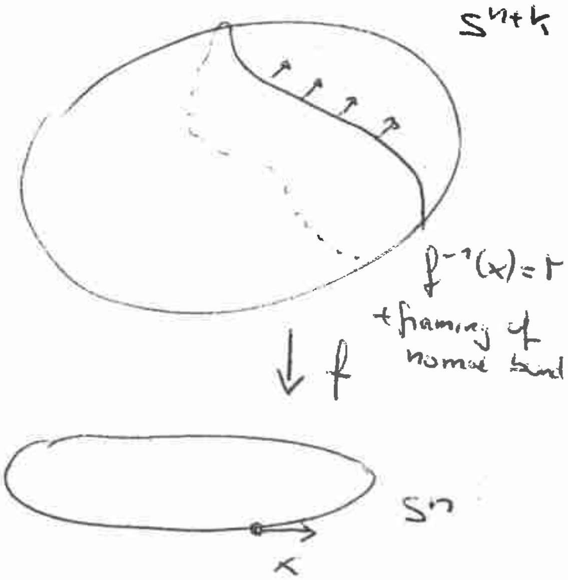
13.10.08
Mike Hopkins

TMF: Overview

Degree of a map: # points in inverse image

-1930's Pontryagin:



$$\pi_{n+k} S^n \hat{=} \text{ framed } k\text{-manifolds}$$

"Framing" is not very geometric structure.

framing → Spin-structure

stable homotopy groups → image of J

1960's Spin-structure ⟷ image of J

Atiyah-Singer index theory

K-local homotopy theory (Adams' e-invariant)

Some stable homotopy grps of spheres. $\pi_k^{st} S^0$

$k=0$	1	2	3	4	5	6	7	8	9	10	11	12	13
$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/6$	$\mathbb{Z}/504$	0	$\mathbb{Z}/3$
								$SU(3)$	$U(3)$	$Sp(2)$			$Sp(1)\times Sp(1)$

framed manifolds not detected by "geometry" →

not detected by "geometric" invariants

14	15
$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/480$
G_2	$U(1)\times G_2$

tmf will explain this portion of stable homotopy grps (although not entirely geometric).

Eisenstein series: $\sigma_k(n) = \sum_{d \mid n} d^k$

$E_2 = 1 - 24 \cdot \sum \sigma_1(n) \cdot q^n$ conf: π_3^s

$E_4 = 1 + 240 \cdot \sum \sigma_3(n)\, q^n$ conf: π_7^s

$E_6 = 1 - 504 \cdot \sum \sigma_5(n) \cdot q^n$ conf: π_{11}^s

$E_8 = 1 + 480 \cdot \sum \sigma_7(n) \cdot q^n$ conf: π_{15}^s

} come up from Bernoulli numbers

1970's: Quillen, Morava: formal groups $\longleftrightarrow$ complex cobordism

$\Downarrow$ Adams-Novikov SS

$\pi_k^s S^0$

Miller, Ravenel, Wilson: Chromatic filtration

$K(n)$ localized homotopy theory

$n=0 \iff$ degree

$n=1 \iff$ K-local homotopy

$n=2 \iff$ next "new" piece of $\pi_k^s S^0$

Chromatic picture relates automorphisms groups of formal grps to homotopy groups

$\mathrm{Aut}(f_g) \longleftrightarrow$ homotopy groups

p-adic Lie groups

$n=1$: p-adic units $\mathbb{Z}_p^\times$

$n=2$: p-adic quaternion algebra

1980's: Ochanine genus

M oriented $\longmapsto$ level 2 modular form

Witten-related Ochanine genus $\to$ geometry on loop spaces

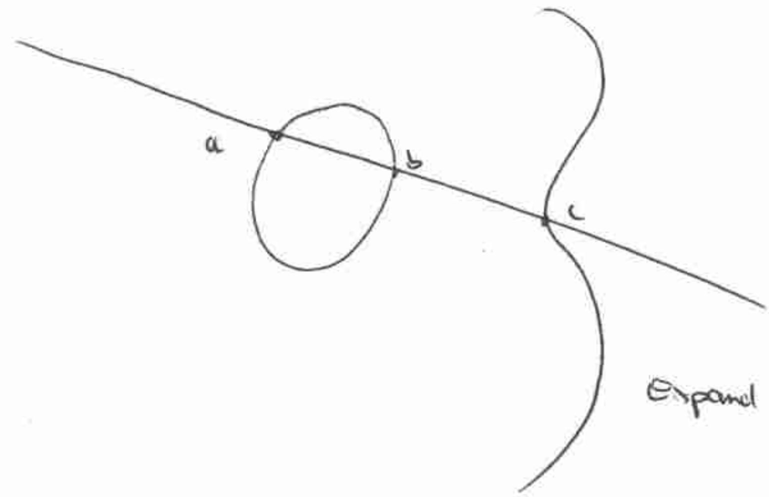
M spin, $\frac{p_1}{2} = 0 \longmapsto$ level 1 modular form

"Wittengenus"

Landweber-Ravenel-Stong: elliptic cohomology

1990s: Hopkins, Mahowald, Miller

Elliptic curve: $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$



group structure by declaring

$a+b+c = e$
$= \text{point at } \infty$

Expand as power series
$\Rightarrow$ formal group

Example: $C: y^2 + y = x^3$ over $\mathbb{F}_4$ has $\# \mathrm{Aut}(C) = 24$ finite !
whereas $\mathrm{Aut}(\text{Formal group}) = $ 2-adic quaternions are infinite.

tmf : relationship between elliptic curves and homotopy theory.

Functor : $\begin{pmatrix}\text{category of} \\ \text{elliptic curves}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{category} \\ \text{of spectra}\end{pmatrix}$

The main issue is rigidification (lot of talks will be spent on this)
Relation to geometry : $MSpin \to KO$

$MString \to tmf$

String = 6-connected cover of Spin
BString = $BO\langle 8\rangle$

14.10.03

Elliptic Curves

Weierstrass cubic $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$

Has a group structure in which colinear points add up to 0.
The point at infinity is the additive unit. This is the "local form"

Scaling $\left.\begin{array}{l} x \mapsto \lambda^{-2}x \\ y \mapsto \lambda^{-3}y \end{array}\right\}$ sends $a_i \mapsto \lambda^i a_i$
+ multiply by λ^6

translations $x \mapsto x + r$

$y \mapsto y + sx + t$

Over Spec(Ring), these local equations glue to "global Weierstraß cubics"

$G = G(r, s, t, \lambda)$ = group of symmetries

Want to make a moduli "space" of Weierstraß elliptic curves

$\mathbb{A}^5 / G$

$(a_1, a_2, a_3, a_4, a_6)$

not a good group action (not free)
$\rightarrow$ bad space

instead we study the stack

$\mathbb{A}^5 /\!/ G = \mathcal{M}_{weier}$

moduli stack of Weierstraß cubics

The stack $\mathcal{M}_{weir}$ associates to every ring the groupoid of all global Weierstraß cubics

$\mathcal{M}_{weir}$: rings (= affine schemes) $\longrightarrow$ groupoids

sheaf of groupoids with "effective descent".

(intrinsic characterization of "global Weierstraß curves":
proper maps of relative dimension 1, all of whose geometric fibres have ~~ge~~ arithmetic genus 1)

Embed : affine schemes $\hookrightarrow$ Stacks
(= com. rings ^{op})

$X \longmapsto (-, X)$ "Yoneda embedding"

Then Weierstraß cubic $/R$ $\Leftrightarrow$ Spec $R \longrightarrow \mathcal{M}_{Weir}$ stack morphism w/ C = Weierstraß curve,

- non-singular — elliptic curve
- singularity :

	nodal sing.	cusp
Ex:	$y^2 - xy = x^3$	$y^2 = x^3$
Formal groups:	$\hat{\mathbb{G}}_{mult}$	$\hat{\mathbb{G}}_{add}$

$\mathcal{M}_{Weier} \setminus \{$cuspidal singularities$\} = \mathcal{M}_{ell}$ moduli stack of generalized elliptic curves.

Modular forms

$A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$, G = group with affine coordinate ring $\mathbb{Z}[r, s, t, \lambda^{\pm 1}]$

What are G-invariant elements of A? Only the constants.
Better: what are the λ^n-eigenspaces in A ?

We can grade A by assigning degrees $|a_i| = 2i$, so that
$$A_{2n} = \lambda^n\text{-eigenspace of } A.$$
Question: what are the (r,s,t)-invariants in A ?

Definition: A modular form of weight n is an element of A_{2n} which is invariant under the (r,s,t)-transformations.

Tate calculated these modular forms.

- Invert 6, then there exist unique (r,s,t) such that the equation becomes
$$y^2 = x^3 + b_2 x^2 + b_4 x + b_6$$
- then substitute $x \mapsto x - b_2/3$ to get the form
$$y^2 = x^3 + \frac{c_4}{48} x + \frac{c_6}{864} \qquad (\text{? sign ?})$$

In fact, $c_4, c_6 \in A$ (i.e. no denominator involving 6)

The expressions c_4 and c_6 are invariant under (r,s,t). Moreover, $c_4^3 - c_6^2$ is divisible by 1728, so

$$\Delta = \frac{c_4^3 - c_6^2}{1728} \in A.$$

Moreover, $$\mathcal{M}_* = \underset{\text{in } A}{(r,s,t)\text{-invariants}} = \mathbb{Z}[c_4, c_6, \Delta] \Big/ c_4^3 - c_6^2 = 1728\,\Delta$$

$(A, \Gamma = A[r,s,t])$ is a Hopf-algebroid and $\mathcal{M}_* = \mathrm{Ext}^{0}_{(A,\Gamma)}(A,A)$.

$$\begin{tikzcd} C \arrow[d] \\ \operatorname{Spec} R \arrow[u, bend left, "e"] \end{tikzcd}$$

"global" Weierstraß cubic
e: section at ∞
– never singular –

pull back tangent bundle at e
$\Rightarrow e^* \Omega^1_C$: a line bundle over Spec R
$=: \omega$

These are functorial to give a line bundle ω over $\mathcal{M}_{Weier}$.

Modular form of weight k = section of ω^k over $\mathcal{M}_{Weier}$.

Let E be a Landweber spectrum (as in H. Miller's talk)

$E^0(\mathbb{C}P^\infty)$ = ring of functions on the formal group G_E.

$$\begin{tikzcd} E^0(\mathbb{C}P^\infty) \arrow[d, "\varepsilon"] \\ E^0(pt) \end{tikzcd} \qquad \begin{tikzcd} f \arrow[d, mapsto] \\ f(0) \end{tikzcd}$$

Thus reduced cohomology $\tilde{E}^0(\mathbb{C}P^\infty)$ = ideal of functions vanishing at $0 =: I$

$\Longrightarrow \pi_2 E = \tilde{E}^0(\mathbb{C}P^1) = I/I^2$ = Zariski cotangent space to G_E at 0

More generally, $\pi_{2n} E = (\Omega^1_{G,0})^{\otimes n} = \omega_G^{\otimes n}$

We want a functor

$$\begin{tikzcd} \operatorname{Spec} R \arrow[d, "C"] \\ \mathcal{M}_{Weier} \end{tikzcd} \rightsquigarrow \mathcal{O}^{top}(C) = \text{a spectrum (Landweber spectrum)}$$

étale —
$\Rightarrow$ the formal group is Landweber exact!

such that $\pi_0\, \mathcal{O}^{top}(C) = R$

formal group of C = formal group of $\mathcal{O}^{top}(C)$

This forces $\pi_{2n}\mathcal{O}^{top}(C) = \omega^{2n}|R$

In other words, $\mathcal{O}_{\mathcal{M}_{Weier}} \cong \pi_0(\mathcal{O}^{top})$ as sheaves on $\mathcal{M}_{Weier}$.

<u>Theorem</u> There exists a sheaf of A_∞ ring spectra on $\mathcal{M}_{Ell}$ called $\mathcal{O}^{top}$ with properties:

- the homology theory underlying $\mathcal{O}^{top}(\operatorname{Spec} R \xrightarrow{C} \mathcal{M}_{Ell})$ is E_C, the Landweber exact homology theory associated to the formal group law of C.

<u>Definition</u>: $$tmf = \mathcal{O}(\mathcal{M}_{Ell})[0,\infty)$$

By associating
$$\begin{array}{c} X \\ \downarrow \text{étale} \\ \mathcal{M}_{Ell} \end{array} \rightsquigarrow \mathcal{O}^{top}(X) \rightsquigarrow \pi_{2n}\mathcal{O}^{top}(X)$$
is a <u>presheaf</u>. We use the notation $\pi_{2n}\mathcal{O}^{top}$ for the associated sheaf.

Then $\pi_{2n}\mathcal{O}^{top} = \omega^n$ and $\pi_{2n+1}\mathcal{O}^{top} = 0$

We get a spectral sequence

$$H^s(X, \pi_t\mathcal{O}^{top}) \Rightarrow \pi_{t-s}\mathcal{O}^{top}(X)$$

or:

$$H^s(X, \omega^n) \Rightarrow \pi_{2n-s}\mathcal{O}^{top}(X)$$

In particular:

$$H^s(\mathcal{M}_{Weier}, \omega^n) \Rightarrow \pi_{2n-s}\, tmf$$
$$\cong$$
$$\mathrm{Ext}^s_\Gamma(A, A)$$

Approaches : • pure obstruction theory (works well away from 2)
determine the htp type of the space of all $\mathcal{O}^{top}$'s
These uses the methods provided by Paul Goerss.

• mixture of construction and obstruction theory
first build tmf "by hand", not knowing may good properties.
then consider

$$\begin{tikzcd} (A_\infty\text{-elliptic spectra}/tmf) \arrow[d, "\operatorname{Spec}\pi_0"] \\ (\mathcal{A}ff/\mathcal{M}_{Weier})_{et} \end{tikzcd} \qquad (tmf \to E)$$

turns out to be an equivalence of categories !

Also uses the obstruction t

This approach works best at $p=2$.

15.10.03

We want $\mathcal{O}^{top}$ - sheaf of A_∞/E_∞ ring spectra on $\mathcal{M}^{ét}_{Ell}$

The construction breaks up into several steps.

① Arithmetic square :

$$\begin{tikzcd} \mathcal{O}^{top} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_p \mathcal{O}^{top}_p \arrow[d] \\ \mathcal{O}^{top}_{\mathbb{Q}} \arrow[r] & \left(\prod_p \mathcal{O}^{top}_p\right)_{\mathbb{Q}} \end{tikzcd}$$

② Hasse square Note : $\mathcal{O}^{top}_p = L_{(K(1)\vee K(2))}\mathcal{O}^{top}$

$$\begin{tikzcd} L_{K(1)\vee K(2))}\mathcal{O}^{top} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L_{K(2)}\mathcal{O}^{top} \arrow[d] \\ L_{K(1)}\mathcal{O}^{top} \arrow[r] & L_{K(1)}L_{K(2)}\mathcal{O}^{top} \end{tikzcd}$$

← completely controlled be Jomel Guys

K(2)- local is (...)

We exploit that
- $K(2)$ local homotopy theory can be controlled
- $L_{K(2)} \mathcal{O}^{top}$ is completely controlled by formal groups.

MP = complex cobordism made 2-periodic $= \bigvee_{n \in \mathbb{Z}} \Sigma^{2n} MU$

$$L_{K(1)} MP = \varprojlim_n \left(v_1^{-1} MP/p^n \right)$$

If M is any MP-module, then similarly, $L_{K(1)} M = \varprojlim_n \left(v_1^{-1} M/p^n \right)$

$$\begin{array}{ll} \operatorname{Spec} R & \\ C \downarrow & \text{elliptic curve such that } \hat{C} \text{ is a Landweber exact formal grp.} \\ \mathcal{M}_{ell} & \text{Then } \mathcal{O}^{top}(C) \text{ is a Landweber exact homology theory with } K(1)\text{-localization as above.} \end{array}$$

<u>Definition</u>: An elliptic curve C over a $\mathbb{Z}/p$-algebra is <u>ordinary</u> if the formal grp has strict height 1. If the strict height is 2, then C is super-singular.

Criterion: if C is defined by Weierstraß equation $f(x,y)=0$. Then C is ordinary $\iff$ coefficient of $(xy)^{p-1}$ in $f(x,y)^{p-1}$ is a unit.

<u>Example</u>: $y^2 = x^3 - x$ in characteristic 2 is super-singular.

$y^2 + xy = x^3 + 1$ in characteristic 2 is ordinary.

$y^2 = x^3 + c_4 x + c_6$ in characteristic $\neq 2, 3$,

$\Rightarrow$ $H(c_4, c_6)$ Hasse polynomial vanishes
(v_1) $\iff$ super-singular.

E.g. for $p=11$, $H(c_4, c_6) = c_4 c_6$.

From now on, everything is p-complete. Let $\mathcal{M}_{ell}^{ord} \subseteq \mathcal{M}_{ell}$ be the substack of those curves whose mod-p reduction is ordinary.

Set $\mathcal{O}^{top}_{ord} = i_*\left(\mathcal{O}^{top}\big|_{\mathcal{M}^{ord}_{Ell}}\right)$. Then $L_{K(1)}\mathcal{O}^{top} = \mathcal{O}^{top}_{ord}$

where $i_* : \mathcal{M}^{ord}_{Ell} \hookrightarrow \mathcal{M}_{Ell}$.

Study $\mathcal{O}^{top}(\mathcal{M}^{ord}) = L_{K(1)}\mathcal{O}^{top}(\mathcal{M}_{Ell}) = L_{K(1)}\, tmf$.

$\underline{p=2}$: every ordinary elliptic curve can be put into the form

$$y^2 + xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

Use $x \mapsto x+r$ to get rid of the term $a_3 y$ (unique r!)

Use $y \mapsto y+t$ to get rid of the term $a_4 x$ (unique t!)

Then under $y \mapsto y + sx$ in characteristic 2 (i.e. $2R=0$), gives

$$y^2 + xy = x^3 + a_2 x^2 + a_6$$

$$\rightsquigarrow \qquad \begin{matrix} s^2 x^2 \\ +sx^2 \end{matrix}$$

After étale extension, can get rid of $a_2 x^2$-term.
So then we have the form

$$y^2 + xy = x^3 + \alpha$$

with universal isomorphism $y \mapsto y + sx$ with $s^2 + s = 0$ (which fixes α).
(this is still true over $\mathbb{Z}_2^\wedge$, not just $\mathbb{Z}/2$).

The universal curve, ordinary, thus lives over $\mathbb{Z}_2[\alpha]$ = 2-completed polynomial ring
with universal automorphism living over $\mathbb{Z}[\alpha][s]/(s^2+s) \cong \mathrm{Hom}_{Set}(\mathbb{Z}/2, \mathbb{Z}_2[\alpha])$

The two solutions $s=0$, $s=-1$ correspond to identity and inverse map on the elliptic curve C.

By Landweber's criterion, $MP_0(-) \otimes_{MP_0} \mathbb{Z}_2[\alpha]$ is a homology theory.
Additively, this $K \otimes \mathbb{Z}_2[\alpha]$ (but it is only a "form" of K-theory multiplicatively).

The $\mathbb{Z}/2$-action is -1 on π_2 of this theory (so looks like ψ_{-1} = complex conjugation). So we get the candidate:

$$L_{K(2)} tmf = KO[\alpha] \quad \text{with a "funny" multiplication.}$$

There is a similar discussion at odd primes.

$K(2)$-local E_∞ ring spectra

R: $K(2)$-local E_∞

Given $\alpha : S^0 \longrightarrow R$, take symmetric (= extended!) powers

$$\begin{tikzcd}
\mathrm{Sym}^p(S^0) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Sym}^p(R) \arrow[r, "E_\infty\text{-structure}"] & R \\
B\Sigma_p^+ \arrow[urr, "\mathrm{Sym}^p(\alpha)"'] & &
\end{tikzcd}$$

$K(2)$-localize everything. Then $B\Sigma_p^+ \xrightarrow{(\varepsilon, \mathrm{tr})} S^0 \times S^0 \simeq S^0 \vee S^0$ is a K-theory equivalence.

Define $\psi : S^0 \longrightarrow B\Sigma_p^+$ by $\varepsilon\psi = 1$, $\mathrm{tr} \circ \psi = 0$
and $\theta : S^0 \longrightarrow B\Sigma_p^+$ by $\varepsilon\theta = 0$, $\mathrm{tr} \circ \theta = 1$.

So cupping with ψ and θ give two operations

$$\psi(\alpha) := \mathrm{Sym}^p(\alpha) \circ \psi \quad \text{and} \quad \theta(\alpha) := \mathrm{Sym}^p(\alpha) \circ \theta$$

These operations satisfy a relation:

$$\begin{tikzcd}
(S^0)^{\wedge p} \arrow[r] \arrow[drr, "\alpha^p"'] & \mathrm{Sym}^p S^0 \arrow[r, "\mathrm{Sym}^p(\alpha)"] & \mathrm{Sym}^p R \arrow[d] \\
& & R
\end{tikzcd}$$

Composed to $B\Sigma_1^+ \overset{t}{\hookrightarrow} B\Sigma_p^+$, so $\varepsilon \circ t = 1$ and $\mathrm{tr} \circ t = p$, thus $t = \psi + p\theta$,
which yields

$$\boxed{\psi(\alpha) + p\,\theta(\alpha) = \alpha^p}$$

Also check: ψ is a ring homomorphism.

The operation θ was first introduced by McClure.

Set $\mathrm{Sym}(S^0)$ = free $K(2)$-local E_∞-ring spectrum on S^0.

Thm (McClure): $R_* \mathrm{Sym}(S^0) = R_* [x, \theta x, \theta^2 x, \dots]$
is the free θ-algebra on one variable

(θ-algebra: ring with homomorphism ψ and a map θ such that
$\psi(\alpha) + p\theta(\alpha) = \alpha^p$)

Example: K_p^1 is a $K(2)$-local E_∞-ring
But $K_p^1[\zeta_p]$ (adjoin pth root of 1) is not E_∞
since there is no map $\psi: \mathbb{Z}_p[\zeta_p]$ such that $\psi(x) = x^p \bmod (p)$

Elliptic curve $C/_R$ for R p-complete, and C ordinary mod p.
What are points of order p in C? $(x,y) \in C$ with $[p](x,y) = 0$.
This grp looks like $\mathbb{Z}/p \times \mathbb{Z}/p$ (after étale extension ...)
Ordinary: one of the $\mathbb{Z}/p$-factors has coordinates divisible by p,
i.e. $\ker(\times p): C(R \otimes \mathbb{Z}/p)\} = \mathbb{Z}/p$.
So there is a distinguished subgrp of C of order p
(really an algebraic subgrp)

$$0 \to {}_pC^{1} \rightarrowtail {}_pC \to {}_pC^{et} \to 0$$

So get a morphism

$$\mathcal{M}^{ord} \longrightarrow \mathcal{M}^{ord}$$
$$C \longmapsto C/\text{canonical subgrp of order } p$$

This determines the "funny" multiplication and the ψ, θ operation on $KO[\![x]\!]$.

In ~~Weierstrass~~ coordinates: $\alpha \mapsto \alpha^p$.

Today we ... $p=2$, the candidate ... This is ... Recall