Celestial Mechanics

(especially central configurations)

Lecture Notes by:

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Celestial mechanics is the study of point particles in $\mathbb{R}^3$ moving under the influence of their mutual gravitational attraction.

A point particle is characterized by a position

$$q = (x, y, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

and a mass $m \in \mathbb{R}^+$. A motion of such a particle is a smooth curve $q(t)$ where $t \in \mathbb{R}$ represents time. Then one defines:

$$\text{Velocity:} \quad v(t) = \dot{q}(t) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \in \mathbb{R}^3$$

$$\text{Momentum:} \quad p(t) = mv(t) = m\dot{q}(t) \in \mathbb{R}^3$$

Newton's equation of motion is

$$\dot{p} = F(q,t) \qquad \text{or} \qquad m\ddot{q} = F(q,t)$$

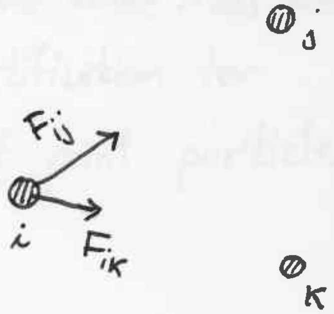
where $F(q,t)$ is the force acting on the particle.

Consider n point particles with positions $q_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$ and masses m_i; $i = 1, \dots, n$. Introduce the notation

$$q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \in \mathbb{R}^{3n} \qquad v = \dot{q} \qquad p = Mv \qquad M = \underbrace{\begin{bmatrix} m_1 & & & & & & \\ & m_1 & & & & O & \\ & & m_1 & & & & \\ & & & \ddots & & & \\ & O & & & m_n & & \\ & & & & & m_n & \\ & & & & & & m_n \end{bmatrix}}_{3n} 3n$$

According to Newton, the gravitational force acting on particle i due to the presence of particle j is

$$F_{ij} = \frac{m_i m_j (q_j - q_i)}{|q_j - q_i|^3}$$



and the total force on particle i is

$$F_i = \sum_{j \neq i} F_{ij}$$

This can be written

$$F_i = \nabla_i U(q)$$

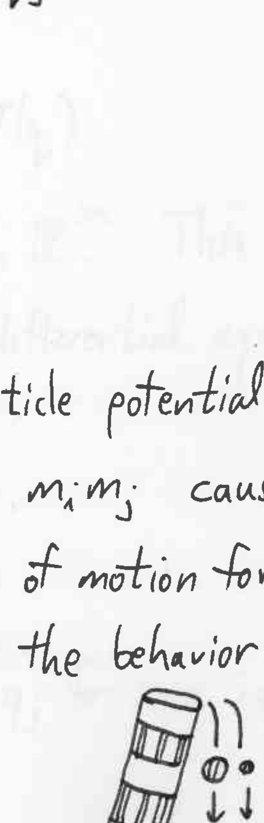
where

$$U(q) = \sum_{\substack{(i,j) \\ i<j}} \frac{m_i m_j}{|q_i - q_j|} \qquad \text{(Newtonian potential)}$$

and

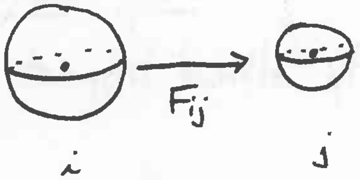
$$\nabla_i = \begin{bmatrix} \frac{\partial}{\partial x_i} \\ \frac{\partial}{\partial y_i} \\ \frac{\partial}{\partial z_i} \end{bmatrix}.$$

Note: The Newtonian interparticle potential has two important properties. First, the factor $m_i m_j$ causes the mass m_i to cancel out of the equation of motion for q_i. This agrees with Galileo's observation that the behavior of a falling body is independent of its mass



Second, the function $\frac{1}{|q_i - q_j|}$ is a harmonic function of $q_i \in \mathbb{R}^3$.

It follows (Gauss) that the potential due to a spherically symmetric mass distribution is the same as if the whole mass were concentrated at the center. This is some justification for the use of point particles.



The equations of the Newtonian n-body problem are:

$$\dot{p}_i = m_i \ddot{q}_i = \nabla_i U(q) \qquad i = 1, \cdots, n$$

or

$$\dot{p} = M \ddot{q} = \nabla U(q)$$

where ∇ is the gradient in $\mathbb{R}^{3n}$. This is a system of real analytic, second order differential equations on the configuration space

$$X = \mathbb{R}^{3n} \setminus \Delta$$

where

$$\Delta = \{ q : q_i = q_j \text{ for some } i \neq j \}$$

is the collision set.

This is equivalent to the first order system

$$\dot{q} = v$$
$$\dot{v} = M^{-1}\nabla U(q)$$

on the *phase space* $TX = \{(q,v) : q \in X, v \in \mathbb{R}^{3n}\} \subset \mathbb{R}^{6n}$.
(TX is the tangent bundle of X.)

Variational Formulations : Introduce a Lagrangian function

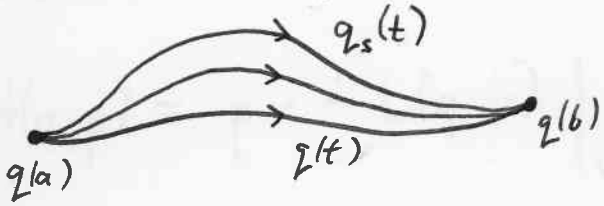
$L : TX \to \mathbb{R}$

$$L(x,v) = \tfrac{1}{2} v^T M v + U(x).$$

If $q(t)$ is a smooth curve define the action of q on $[a,b]$

$$\mathcal{A} = \int_a^b L(q(t), \dot{q}(t))\,dt.$$

The principle of least action states that if $q(t)$ is a motion of the Lagrangian system then $\mathcal{A}$ is stationary under fixed-endpoint variations of q:



$$\delta\mathcal{A} = \frac{d}{ds}\Big|_{s=0} \mathcal{A}(q_s(t)) = 0.$$

The calculus of variations shows that this condition implies the Euler-Lagrange equations. Introduce the conjugate momentum:

$$p = \frac{\partial L}{\partial v}(q,v) = \left[\frac{\partial L}{\partial v_1} \quad \cdots \quad \frac{\partial L}{\partial v_{3n}}\right] \in \mathbb{R}^{3n*}$$

1-forms or covectors

$$p(t) = \frac{\partial L}{\partial v}(q(t),\dot{q}(t))$$

Then the Euler-Lagrange equation is:

$$\dot{p}(t) = \frac{\partial L}{\partial q}(q(t),\dot{q}(t))$$

For the n-body problem Lagrangian, this becomes

$$\dot{v}^T M = DU(q)$$

which is the transpose of Newton's equation.

Solving the equation defining the conjugate momentum for v

$$v = M^{-1} p^T$$

one can define a Hamiltonian function $H : T^*X \rightarrow \mathbb{R}$

$$H(q,p) = pv - L(q,v) \Big|_{v = M^{-1}p^T}$$

$$= \frac{1}{2} p M^{-1} p^T - U(q)$$

Differentiating the definition of H with respect to p gives

$$\frac{\partial H}{\partial p} = v + \left(\cancelto{0}{p - \frac{\partial L}{\partial v}(q,v)}\right) \frac{\partial v}{\partial p}\bigg|_{v = M^{-1}p} = v = \dot{q}$$

The Euler-Lagrange equation is

$$\dot{p} = -\frac{\partial H}{\partial q}.$$

Thus Hamilton's equations hold:

$$\dot{q} = \frac{\partial H}{\partial p}(q,p) \quad \leftarrow \text{vectors} \quad \left(\text{Note: } \frac{\partial H}{\partial p} \in \left(\mathbb{R}^{3n*}\right)^* \simeq \mathbb{R}^{3n}\right)$$

$$\dot{p} = -\frac{\partial H}{\partial q}(q,p) \quad \nwarrow \text{1-forms}$$

Since $L(q,v) = pv - H(q,p)\big|_{p = v^T M}$, the action is

$$\mathcal{Q} = \int_a^b \left[p(t)\,\dot{q}(t) - H(q(t), p(t))\right] dt$$

If one forgets the definition of p in terms of the Lagrangian, this can be viewed as a functional of the curve $(q(t), p(t))$ in T^*X rather than the curve $q(t)$ in X. Hamilton's equations follow from assuming this curve is stationary under variations in T^*X which fix the end points

The variational approach facilitates changes of coordinates. Let (Q, P) denote new coordinates related to the old ones by a diffeomorphism

$$q = q(Q,P)$$
$$p = p(Q,P)$$

Let

$$K(Q,P) = H(q(Q,P), p(Q,P))$$

and suppose that

$$p(Q(t),P(t))\,\dot{\overline{q(Q(t),P(t))}} = P(t)\dot{Q}(t)$$

for every curve $(Q(t),P(t))$. Another way to state this is to require equality of differential forms

$$p(Q,P)\,d\,q(Q,P) = P\,dq$$

or, less formally,

$$p\,dq = P\,dQ.$$

Then the action integrals are equal on corresponding curves:

$$\int_a^b \left[p(t)\dot{q}(t) - H(q(t),p(t))\right]dt = \int_a^b \left[P(t)\dot{Q}(t) - K(Q(t),P(t))\right]dt$$

if $(q(t), p(t)) = (q(Q(t),P(t)), p(Q(t),P(t)))$.

It follows that Hamilton's equations hold in the new coordinates

$$\dot{Q} = \frac{\partial K}{\partial P}$$

$$\dot{P} = -\frac{\partial K}{\partial Q}.$$

As an example, consider Jacobi coordinates for the 3-body problem. Introduce new coordinates

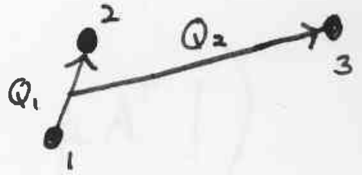
$$\bar{q} = \frac{1}{\bar{m}}(m_1 q_1 + m_2 q_2 + m_3 q_3) \quad \text{center of mass} \quad , \quad \bar{m} = m_1 + m_2 + m_3$$

$$Q_1 = q_2 - q_1$$

$$Q_2 = q_3 - \gamma_1 q_1 - \gamma_2 q_2 \qquad \gamma_1 = \frac{m_1}{m_1+m_2}, \gamma_2 = \frac{m_2}{m_1+m_2}$$

and conjugate momenta

$$\bar{p} = \frac{1}{\bar{m}} \dot{\bar{q}}^T \qquad P_1 = \alpha \dot{Q}_1^T \qquad P_2 = \beta \dot{Q}_2^T$$



$$\alpha = \frac{m_1 m_2}{m_1 + m_2} \qquad \beta = \frac{(m_1+m_2)m_3}{\bar{m}}.$$

Then $\bar{p}d\bar{q} + P_1 dQ_1 + P_2 dQ_2 = p_1 dq_1 + p_2 dq_2 + p_3 dq_3$ and the new Hamiltonian is

$$K(Q,P) = \frac{1}{2}\bar{m}^{-1}|\bar{p}|^2 + \frac{1}{2}\alpha^{-1}|P_1|^2 + \frac{1}{2}\beta^{-1}|P_2|^2 - U(Q)$$

$$U(Q) = \frac{m_1 m_2}{|Q_1|} + \frac{m_1 m_3}{|Q_2 + \gamma_2 Q_1|} + \frac{m_2 m_3}{|Q_2 - \gamma_1 Q_1|}.$$

Symmetries and Integrals

The Hamiltonian of the n-body problem is symmetric under the action of the Euclidean group Euc(3). An element $g \in \mathrm{Euc}(3)$ takes the form

$$g \cdot x = Ax + b$$

$A \in O(3)$ 3×3 orthogonal matrix

$b \in \mathbb{R}^3$ translation

$x \in \mathbb{R}^3$

Extend this action to $T^*X \subset \mathbb{R}^{3n} \times \mathbb{R}^{3n*}$ via.

$$g \cdot (q,p) = g \cdot \left(\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}, [p_1 \cdots p_n] \right)$$

$$= \left(\begin{bmatrix} g \cdot q_1 \\ \vdots \\ g \cdot q_n \end{bmatrix}, [p_1 A^T \quad \cdots \quad p_n A^T] \right)$$

Then

$$H(g\cdot(q,p)) = \frac{1}{2} \sum_i m_i^{-1} p_i A^T (p_i A^T)^T - \sum_{i<j} \frac{m_i m_j}{|g\cdot q_i - g \cdot q_j|}$$

$$= \frac{1}{2} \sum_i m_i^{-1} p_i A^T A p_i^T - \sum_{i<j} \frac{m_i m_j}{|A(q_i - q_j)|}$$

$$= H(q,p)$$

since A is orthogonal.

This action also preserves the form $p\,dq$. It follows that

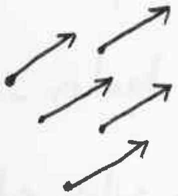
$$(q(t),p(t)) \text{ solves the n-body problem} \iff g\cdot(q(t),p(t)) \text{ does.}$$

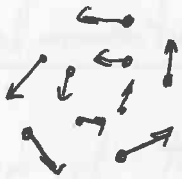
Let g_s be a one-parameter family of Euclidean transformations and suppose $g_0 = I$. Define a vectorfield

$$X(q) = \frac{d}{ds}\, g_s\cdot(q)\,\Big|_{s=0}$$

$g_s x = x + s\cdot b$

$$g_s x = \begin{bmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{bmatrix} x$$





Noether's theorem states that if g_s is a family of symmetries of the Hamiltonian $H(q,p)$ then

$$F(q,p) = p\,X(q)$$

is an integral of Hamilton's equations, that is,

$$F(q(t),p(t)) = \text{const}$$

if $(q(t),p(t))$ is a solution.

Using this, one can derive the classical integrals of the n-body problem.

Let $g_s^x = x + sb$, $b \in \mathbb{R}^3$. Then $g_s \cdot q = \begin{bmatrix} q_1 + sb \\ \vdots \\ q_n + sb \end{bmatrix}$ so

$$X(q) = \begin{bmatrix} b \\ \vdots \\ b \end{bmatrix} \quad \text{and} \quad F(q,p) = [p_1 \cdots p_n] \begin{bmatrix} b \\ \vdots \\ b \end{bmatrix}$$
$$= \left(\sum_{i=1}^{n} p_i \right) b .$$

Since this is an integral for all $b \in \mathbb{R}^3$, the quantity

$$\bar{p} = \sum_{i=1}^{n} p_i \in \mathbb{R}^{3*}$$

is a form-valued integral, called the <u>total momentum</u>.

The value of $\bar{p}$ determines the motion of the <u>center of mass</u>:

$$\bar{q} = \frac{1}{\bar{m}} \sum_{i=1}^{n} m_i q_i \quad \text{where} \quad \bar{m} = \sum_{i=1}^{n} m_i .$$

Clearly $\dot{\bar{q}} = \frac{1}{\bar{m}} \bar{p}$ so $\bar{q}(t) = \bar{q}(t_0) + \frac{1}{\bar{m}} \bar{p}(t - t_0)$.

If $(q(t), p(t))$ is any solution of the n-body problem with momentum $\bar{p}$ then

$$\tilde{q}_i(t) = q_i(t) - \bar{q}(t) \qquad \tilde{p}_i(t) = p_i(t) - \frac{m_i}{\bar{m}} \bar{p}$$

is another solution, but with total momentum $\bar{\tilde{p}} = 0$.

Thus one can study solutions with $\bar{p} = 0$ without loss of generality.

With $\bar{p} = 0$, $\bar{q}$ becomes a vector-valued constant of motion. The solution $\tilde{q}(t)$ has $\bar{\tilde{q}} = 0$ so one may assume

$$\bar{q} = \bar{p} = 0.$$

Further integrals result from applying Noether's theorem to a one-parameter family of rotations with constant angular velocity vector $w \in \mathbb{R}^3$, $g_s x = R_s x$. Then

$$\frac{d}{ds} R_s \Big|_{s=0} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} = W \in so(3)$$

The action of W can be represented using cross products : $W_x = w \times x$

It follows that

$$X(q) = \begin{bmatrix} W q_1 \\ \vdots \\ W q_n \end{bmatrix} = \begin{bmatrix} w \times q_1 \\ \vdots \\ w \times q_n \end{bmatrix}, \quad F(q,p) = \sum_{i=1}^{n} p_i (w \times q_i).$$
$$= \sum_{i=1}^{n} w^T (q_i \times p_i^T)$$

It follows that

$$\Omega = \sum_{i=1}^{n} q_i \times p_i^T \in \mathbb{R}^3$$

is a vector-valued integral. It is called the <u>angular momentum</u>.

The integrals $\bar{p}$, $\bar{q}$, Ω provide 9 constants of motion (at least if $\bar{p}=0$). The Hamiltonian

$$H(q,p) = \tfrac{1}{2}\, p M^{-1} p^T - U(q)$$

is the 10 th.

Integrals give rise to invariant sets in phase space. Given constants ω, h the set

$$S_{(\omega_0,h)} = \{ (q,p) \in T^*X : \ \bar{q}=0, \bar{p}=0, \Omega=\omega_0, H=h \}$$

is invariant. One expects that for most choices of ω, h this will be a $(6n-10)$-dimensional manifold. The conditions for this to be the case will now be investigated. Consider the gradients of the integrals (p will be viewed as a vector instead of a form during these computations):

$$\nabla\bar{q} = \frac{1}{m}\begin{array}{cc} \left[\begin{array}{c} m_1 I \\ \vdots \\ m_n I \\ \hdashline O \\ \vdots \\ O \end{array}\right] & \begin{array}{l} 3n \\ \\ \\ 3n \end{array} \\ 3 & \end{array}, \quad \nabla\bar{p} = \begin{array}{cc} \left[\begin{array}{c} O \\ \vdots \\ O \\ \hdashline I \\ \vdots \\ I \end{array}\right] & \begin{array}{l} 3n \\ \\ \\ 3n \end{array} \\ 3 & \end{array}, \quad \nabla\Omega = \begin{array}{cc} \left[\begin{array}{c} -\tilde{p}_1 \\ \vdots \\ -\tilde{p}_n \\ \hdashline \tilde{q}_1 \\ \vdots \\ \tilde{q}_n \end{array}\right] & \begin{array}{l} 3n \\ \\ \\ 3n \end{array} \\ 3 & \end{array}, \quad \nabla H = \begin{array}{cc} \left[\begin{array}{c} -\nabla U(q) \\ \hdashline M^{-1} p \end{array}\right] & \begin{array}{l} 3n \\ 3n \end{array} \\ 1 & \end{array}$$

where

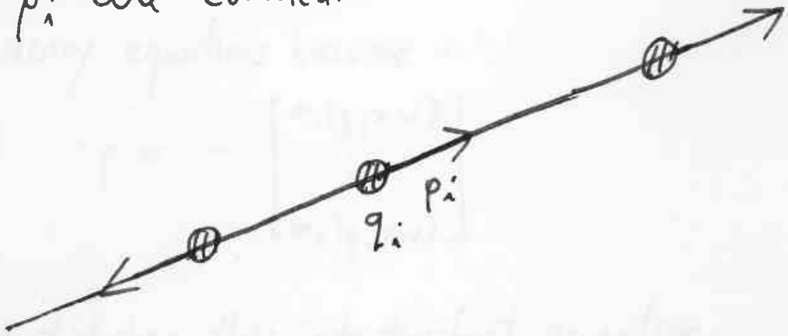
$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{q}_i = \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}, \quad \tilde{p}_i = \begin{bmatrix} 0 & -\zeta_i & \eta_i \\ \zeta_i & 0 & -\xi_i \\ -\eta_i & \xi_i & 0 \end{bmatrix}$$

$$(q_i = (x_i, y_i, z_i)) \qquad (p_i = (\xi_i, \eta_i, \zeta_i))$$

The integrals are independent at (q,p) provided the $(3n\times 10)$ matrix

$$\left[\nabla\bar{q} \,\vdots\, \nabla\bar{p} \,\vdots\, \nabla\Omega \,\vdots\, \nabla H\right] \quad \text{has rank } 10 \text{ at } (q,p).$$

It is not hard to show that the first 9 columns are independent except for points (q,p) when all the q_i and p_i are collinear



With our assumptions that $\bar{q}=\bar{p}=0$ this situation implies $\omega_0=0$ as well. Thus it can be ruled out by assuming $\omega_0\neq 0$.

The only other way for independence to fail is if there exist constant vectors $u,v,w\in\mathbb{R}^3$ with

$$\nabla H = \nabla\bar{q}\,u + \nabla\bar{p}\,v + \nabla\Omega\,w$$

that is

$$-\nabla U(q) = \frac{1}{\bar{m}}\begin{bmatrix} m_1 u \\ \vdots \\ m_n u\end{bmatrix} + \begin{bmatrix} -p_1\times w \\ \\ -p_n\times w\end{bmatrix}$$

$$\text{and} \quad M^{-1}p = \begin{bmatrix} v \\ \vdots \\ v\end{bmatrix} + \begin{bmatrix} q_1\times w \\ \\ q_n\times w\end{bmatrix}$$

Summing the n 3-dimensional components of the first equation gives

$$0 = u - \bar{p} \times w = u \Rightarrow u = 0$$

(The left side vanishes because $F_{ij} = -F_{ji}$)

Summing in the second equation with weights m_i gives

$$\bar{p} = \bar{m} v + \cancelto{0}{\overline{mq}} \times w \Rightarrow v = 0 .$$

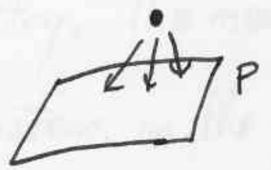
The dependency equations become:

$$p = - \begin{bmatrix} m_1(q_1 \times w) \\ \vdots \\ m_n(q_n \times w) \end{bmatrix}$$

and, substituting this into the first equation,

$$\nabla U(q) = \begin{bmatrix} m_1(q_1 \times w) \times w \\ \vdots \\ m_n(q_n \times w) \times w \end{bmatrix} = -|w|^2 \begin{bmatrix} m_1 q_1^{\perp} \\ \vdots \\ m_n q_n^{\perp} \end{bmatrix}$$

where the superscript $\perp$ denotes orthogonal projection onto the plane, P, normal to w. It follows that $\nabla_i U \in P$ for $i = 1, \dots, n$ and from this one finds $q_i \in P$ (otherwise, consider the particle farthest from P to get a contradiction)



Thus the $\perp$ is unnecessary and we find that q satisfies an equation of the form

$$M^{-1} \nabla U(q) + \lambda q = 0 \qquad \lambda > 0 \qquad *$$

Conversely, if $q \in X$ is a planar configuration satisfying $*$ then one can reconstruct $w \in \mathbb{R}^3$ and p so that (q,p) is a point where the 10 integrals are dependent. A configuration q of this type will be called a <u>critical configuration</u>. If $S_{(\omega_0,h)}$ contains no point of the form (q,p) with q a critical configuration and p the corresponding momentum, then $S_{(\omega_0,h)}$ is a smooth $(6n-10)$-dimensional submanifold of T^*X.

Unfortunately, equation $*$ is very difficult to solve. Further discussion of it will be postponed until later, where it will arise in the course of studying a very different question.

One can invoke symmetry one last time to eliminate another dimension. Note that $S_{(\omega_0,h)}$ is still invariant under orthogonal transformations which fix $\omega_0 \in \mathbb{R}^3$. In particular, there is a one parameter rotational symmetry. This means that there is a well-defined dynamical system on the quotient space $S'_{(\omega_0,h)}$, which will be a manifold of dimension $(6n-11)$ if $S_{(\omega_0,h)}$ is a manifold of dimension $(6n-10)$ as above.

Because the Newtonian potential is homogeneous, it is possible to obtain new solutions by scaling. If $(q(t), p(t))$ is a solution in $S_{(\omega_0, h)}$ then for any constant $\sigma \neq 0$,

$$\tilde{q}(t) = \sigma^{-2} q(\sigma^3 t) \qquad \tilde{p}(t) = \sigma p(\sigma^3 t)$$

is a solution in $S_{(\sigma^{-1}\omega_0, \sigma^2 h)}$. This shows that there is essentially only one parameter in the n-body problem (besides the masses) namely: $|\omega_0|^2 h$. By means of this scaling one may normalize the energy, h, to $-1, 0,$ or 1 according to sign.

Another useful rescaling allows one to normalize the masses. If $(q(t), p(t))$ is a solution of the n-body problem for masses $(m_1 \cdots m_n)$ then for any constant $\mu \neq 0$

$$\tilde{q}(t) = q(\mu t) \qquad \tilde{p}(t) = \mu^3 p(\mu t)$$

is a solution for masses $(\mu^2 m_1, \dots, \mu^2 m_n)$. It is common to normalize the masses so that $\overline{m} = \sum_{i=1}^{n} m_i = 1$.

The Two-Body Problem For $n=2$, the reductions described above can be used to pass from the $6n=12$ dimensional phase space to a manifold of dimension $6n-11=1$, which completely solves the problem.

$$X = \{(q_1,q_2) \in \mathbb{R}^6 : q_1 \neq q_2\} \qquad T^*X = \{(q_1,q_2,p_1,p_2)\} \subset \mathbb{R}^6 \times \mathbb{R}^{6*}$$

The equations of motion are

$$\dot{q}_1 = m_1^{-1} p_1^T \qquad \dot{q}_2 = m_2^{-1} p_2^T$$

$$\dot{p}_1^T = \frac{m_1 m_2 (q_2 - q_1)}{|q_2 - q_1|^3} \qquad \dot{p}_2^T = \frac{m_1 m_2 (q_1 - q_2)}{|q_1 - q_2|^3}$$

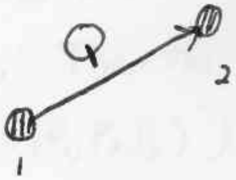
Scale the masses so $\bar{m} = m_1 + m_2 = 1$ and introduce new coordinates

$\bar{q} = m_1 q_1 + m_2 q_2$ = center of mass

$\bar{p} = p_1 + p_2$ = total momentum

$Q = q_2 - q_1$ = relative position

$P = \frac{p_2}{m_2} - \frac{p_1}{m_1}$ = relative velocity



Then the differential equations become

$$\dot{\bar{q}} = \bar{p}^T \qquad \dot{Q} = P^T$$

$$\dot{\bar{p}} = 0 \qquad \dot{P}^T = \frac{-Q}{|Q|^3}$$

Elimination of the total momentum and center of mass are accomplished by ignoring $\bar{q}$ and $\bar{p}$, leaving a Hamiltonian system on $T^*(\mathbb{R}^3 \setminus \{0\}) \simeq (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^{3*}$

$$H(Q,P) = \frac{1}{2}|P|^2 - \frac{1}{|Q|}$$

This is called the Kepler problem. Assuming that $\bar{p}=0$, the angular momentum is

$$\Omega(Q,P) = m_1 m_2 \, Q \times P^T$$

The integral sets are (dropping the $m_1 m_2$ factor)

$$S_{(\omega_0,h)} = \{(Q,P) \in (\mathbb{R}^3 \setminus 0) \times \mathbb{R}^{3*} : Q \times P^T = \omega_0, \, H(Q,P) = h\}$$

Using the rotation symmetry, any $\omega_0 \neq 0$ could be rotated to a vertical vector. So one may assume that $\omega_0 = (0,0,k)$. Then the equation $Q \times P^T = \omega_0$ becomes (with $Q = (Q_1, Q_2, Q_3)$, $P = (P_1, P_2, P_3)$)

$$\left.\begin{aligned} P_3 Q_2 - P_2 Q_3 &= 0 \\ P_1 Q_3 - P_3 Q_1 &= 0 \\ P_2 Q_1 - P_1 Q_2 &= k \end{aligned}\right\} \longmapsto \underbrace{\begin{bmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{bmatrix}}_{\text{determinant } = k.} \begin{bmatrix} Q_3 \\ -P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $k \neq 0$ these imply $Q_3 = P_3 = 0$, that is, the position and momentum are in the (x,y)-plane. This is an invariant set for the equations of motion, so one can eliminate 2 more variables by ignoring Q_3, P_3.

So view $(Q,P) \in T^*(\mathbb{R}^2 \setminus 0) \simeq (\mathbb{R}^2 \setminus 0) \times \mathbb{R}^{2*}$

$$S_{(\omega_0,h)} = \{(Q,P) \in (\mathbb{R}^2 \setminus 0) \times \mathbb{R}^{2*} : P_2 Q_1 - P_1 Q_2 = k, \, H(Q,P) = h\}$$

The rest of the angular momentum reduction is easiest to accomplish in polar coordinates. Introduce new variables (reusing letter ρ)

$$r = |Q| \qquad \tan\theta = Q_2/Q_1$$

$$\rho = \dot{r} = \frac{PQ}{|Q|^2} \qquad \Omega = Q_1 P_2 - P_1 Q_2$$

The equations of motion are

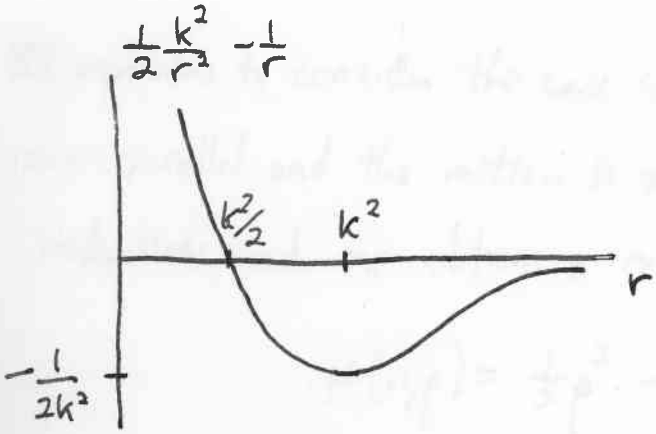
$$\dot{\theta} = \frac{\Omega}{r^2} \qquad \dot{r} = \rho$$

$$\dot{\Omega} = 0 \qquad \dot{\rho} = -\frac{1}{r^2} + \frac{\Omega^2}{r^3}$$

The angular momentum equation is just $\Omega = k$ and then the angle θ represents the symmetry that one would like to eliminate in passing from $S_{(\omega,h)}$ to the quotient space $S'_{(\omega_0,h)}$. Thus fixing Ω and passing to the quotient amounts to ignoring Ω and θ and setting $\Omega = k$ in the $\dot{\rho}$ equation. Thus we have the reduced Hamiltonian system

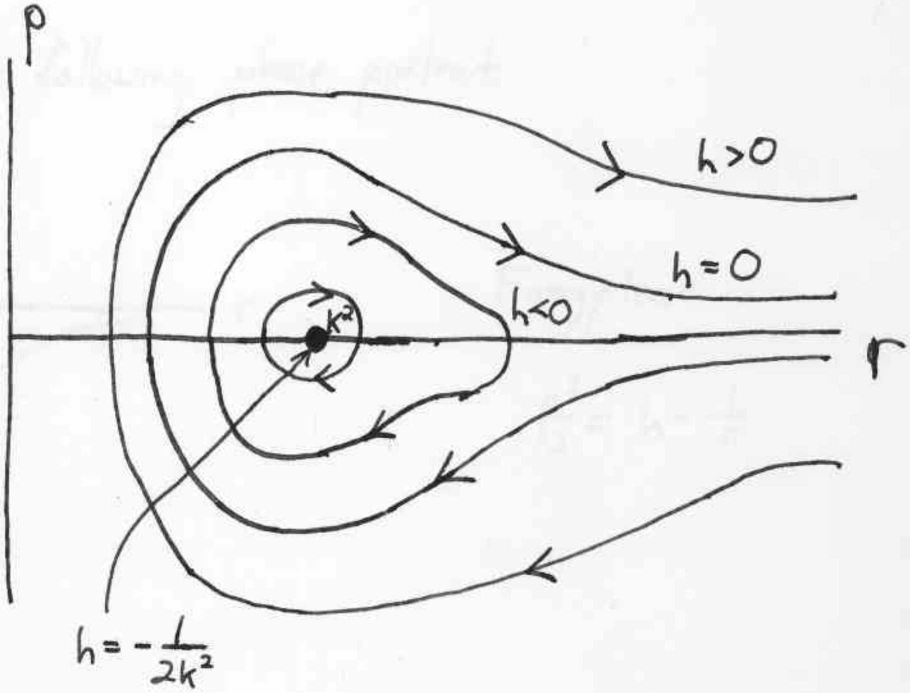
$$H(r,\rho) = \tfrac{1}{2}\rho^2 + \tfrac{1}{2}\frac{k^2}{r^2} - \frac{1}{r}$$

on $T^*\mathbb{R}^+ \simeq \mathbb{R}^+ \times \mathbb{R}$. This one degree of freedom problem can be understood using phase portrait analysis.



Energy levels

$$\frac{p^2}{2} = h - \left(\frac{1}{2}\frac{k^2}{r^2} - \frac{1}{r}\right)$$



Thus the reduced level sets can be classified topologically as:

$S'_{(\omega_0,h)} \simeq \mathbb{R}$ $\quad h|\omega_0|^2 > 0$ or $h=0, |\omega_0| \neq 0$

$S'_{(\omega_0,h)} \simeq S^1$ $\quad -\frac{1}{2} < h|\omega_0|^2 < 0$

$S'_{(\omega_0,h)} \simeq$ point $\quad h|\omega_0|^2 = -\frac{1}{2}$

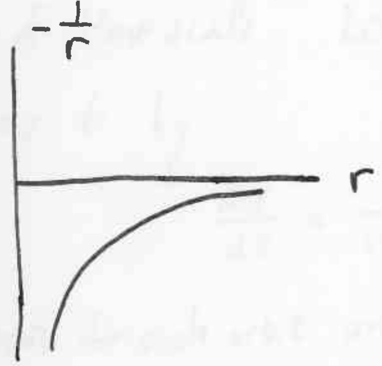
$S'_{(\omega_0,h)} = \emptyset$ $\quad h|\omega_0|^2 < -\frac{1}{2}$

The unreduced sets are either $\mathbb{R}\times S^1$, $T^2 = S^1 \times S^1$, S^1 or $\emptyset$

It remains to consider the case $\omega_0 = 0$. In this case Q and P are parallel and the motion is collinear. One can do the same reductions and one obtains a reduced Hamiltonian

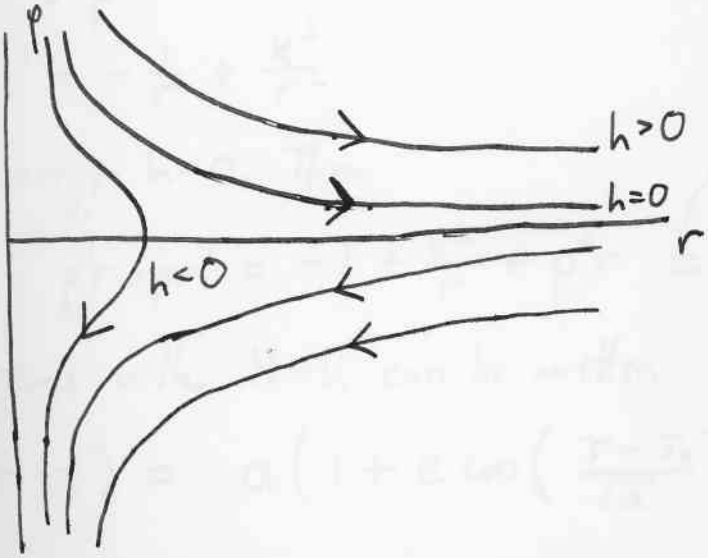
$$H(r,\rho) = \frac{1}{2}\rho^2 - \frac{1}{r}.$$

This leads to the following phase portrait

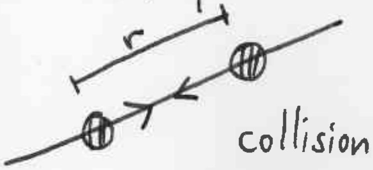


Energy levels

$$\frac{\rho^2}{2} = h - \frac{1}{r}$$



↖ Note : $r \to 0$, $\rho = \dot{r} \to -\infty$



To get a more detailed understanding of the solutions one must find the time parametrization of the orbits, that is, solve

$$\dot{r} = p$$

$$\dot{p} = -\frac{1}{r^2} + \frac{k^2}{r^3} \qquad H = \frac{1}{2}p^2 + \frac{1}{2}\frac{k^2}{r^2} - \frac{1}{r} = h$$

We will consider only the case $h<0$. The equations are not solveable in terms of elementary functions, but can nevertheless be well-understood after a change of time scale. Let τ denote a new parameter related to time t by

$$\frac{d\tau}{dt} = \frac{1}{r(t)}$$

(this is different for each orbit and is not explicitly computable).

Let $'$ denote differentiation with respect to τ. Then

$$r' = pr$$

$$p' = -\frac{1}{r} + \frac{k^2}{r^2}$$

Fix an energy $h<0$. Then

$$r'' = p'r + pr' = -1 + \frac{k^2}{r} + p^2 r \overset{\text{using } H(q,p)=h}{=} 1 + 2hr$$

The solutions with $H=h$ can be written

$$r(\tau) = a\left(1 + e\cos\left(\frac{\tau-\tau_0}{\sqrt{a}}\right)\right)$$

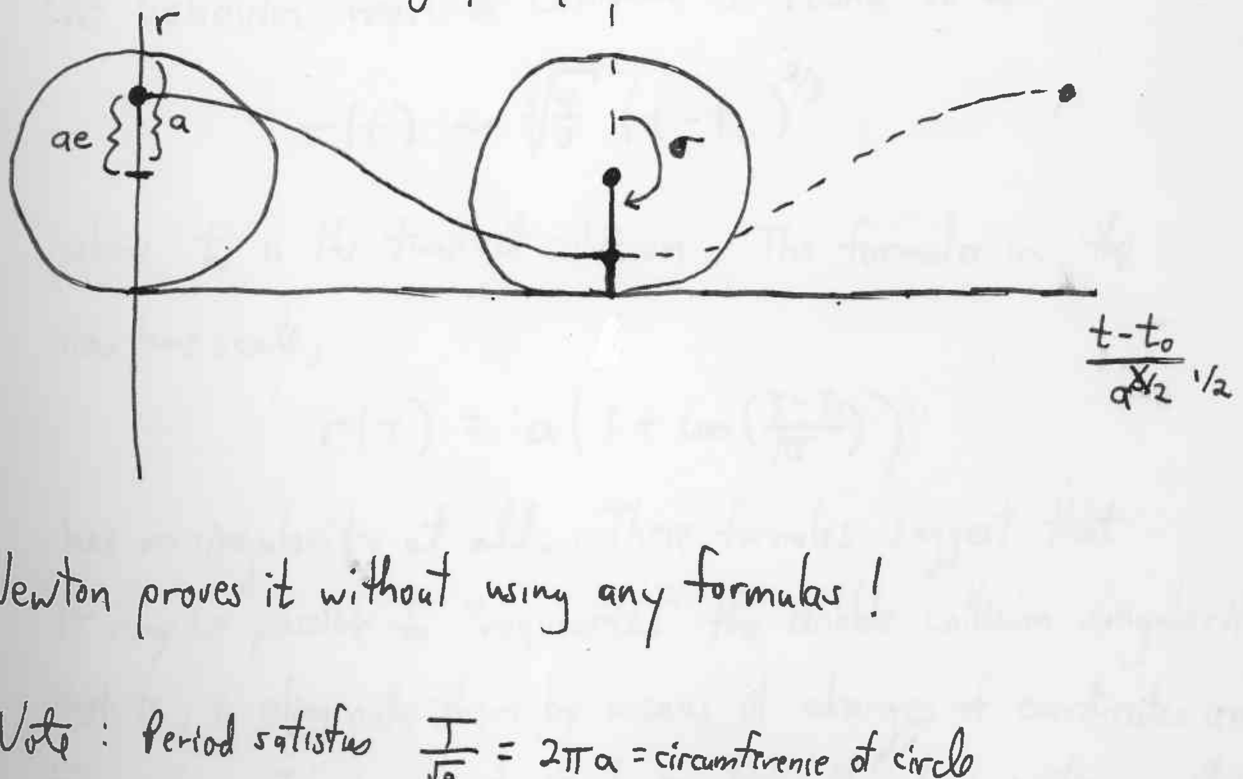
$$a = \frac{1}{2|h|} \qquad e = \sqrt{1-2|h|k^2} \qquad \tau_0 \Leftrightarrow \text{max. radius}$$

Knowing $r(\tau)$, one can integrate $\frac{dt}{d\tau} = r(\tau)$ to find

$$s = \frac{t - t_0}{a^{3/2}} = \sigma + e \sin \sigma \quad \text{where} \quad \sigma = \frac{\tau - \tau_0}{\sqrt{a}}$$

which is Kepler's equation. The inability to invert this equation to get $\tau(t)$ is the only barrier to a complete formula for the solution to the Kepler problem.

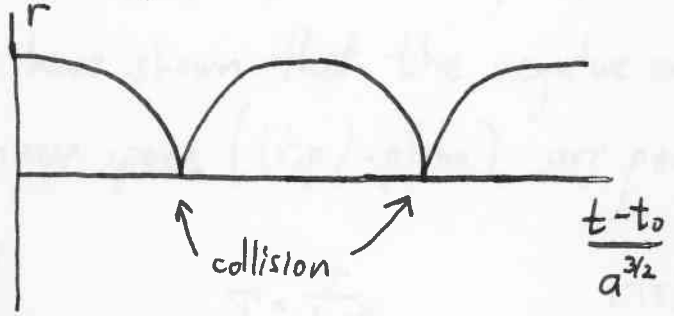
Newton found a geometrical way to construct the graph of $r(s)$. Let a circle of radius a roll without slipping along the s axis in the (s,r) plane and let σ denote the number of radians it has rolled. Then the path of a point at radius ae is the graph of $r(t)$.



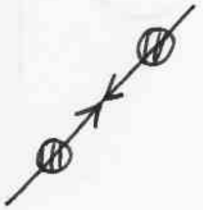
Newton proves it without using any formulas!

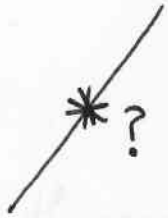
Note: Period satisfies $\frac{T}{\sqrt{a}} = 2\pi a$ = circumference of circle

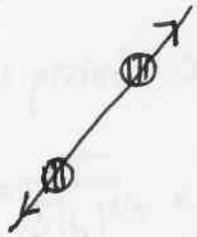
When the angular momentum is $k=0$, $e=1$, $ae=a$ and we obtain a cycloid.



This represents a sequence of double collisions.







The behavior near a collision is found to be

$$r(t) \sim \sqrt[3]{\frac{9}{2}}\,(t-t_c)^{2/3}$$

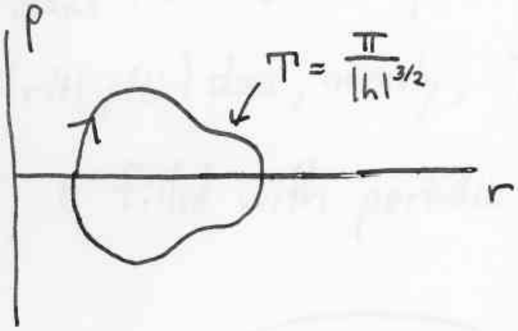
where t_c is the time of collision. The formula in the new time scale,

$$r(s) = a\left(1+\cos\left(\frac{s-s_0}{\sqrt{a}}\right)\right)$$

has no singularity at all. These formulas suggest that it may be possible to "regularize" the double collision singularities, that is, to eliminate them by means of changes of coordinates and timescales. It is important to regularize the whole system, rather than simply studying individual collision orbits, however.

Before proceeding with regularization, it is worthwhile to point out another curious property of the Kepler problem.

We have shown that the negative energy orbits in the reduced phase space ((r,ρ)-plane) are periodic.



$r(\varsigma)$ has period $2\pi\sqrt{a}$

From Newton's figure one finds that

$r(t)$ has period $2\pi a^{3/2}$

where $a = \frac{1}{2|h|}$. Thus the period is $T = \frac{\pi}{\sqrt{2}|h|^{3/2}}$.

Now consider the unreduced integral set $S_{(\omega_0,h)} \simeq T^2$.

The angular variables describing the torus are some angle in the (r,ρ)-plane together with the variable Θ which was ignored during reduction. To understand the dynamics on the torus we need to know

$$\Delta\Theta = \int_0^T \dot{\Theta}(t)\,dt = \int_0^{2\pi\sqrt{a}} \Theta'(\varsigma)\,d\varsigma$$

Recall that Θ satisfies

$$\dot{\Theta} = \frac{k}{r^2} \Rightarrow \Theta'(\varsigma) = \frac{k}{r(\varsigma)}$$

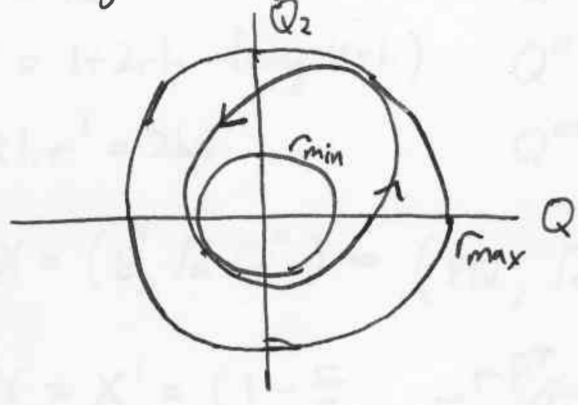
Using the formula for $r(\tau)$ and setting $\sigma = \frac{\tau}{\sqrt{a}}$ gives

$$\Delta\theta = \frac{k}{\sqrt{a}} \int_0^{2\pi} \frac{d\sigma}{1 + e\cos\sigma} = \frac{k}{\sqrt{a}} \frac{2\pi}{\sqrt{1-e^2}} = \pm 2\pi$$

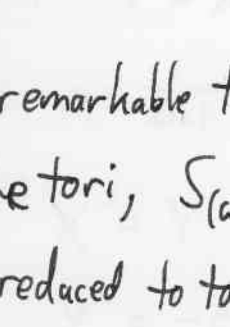
according to the sign of k (recall $e = \sqrt{1-2|h|k^2} = \sqrt{1-\frac{k^2}{a}}$.)

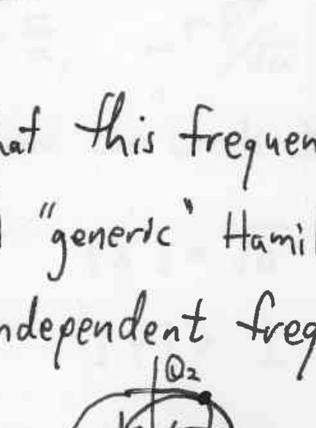
This means that $\theta(t)$ completes one cycle in exactly the same time that $(r(t), p(t))$ does, namely, $T = 2\pi a^{3/2}$. So the torus $S_{(\omega_0, h)}$ is filled with periodic orbits.

In the configuration space (reduced to a plane) we have



The remarkable thing is that this frequency-locking occurs on all of the tori, $S_{(\omega_0, h)}$. A "generic" Hamiltonian with symmetry will reduced to tori with independent frequencies with solution curves typically dense in the torus.





We will now make some coordinate changes which will simultaneously regularize the double collision singularity and give an explanation for the persistent frequency locking.

Returning to the unreduced Kepler problem, introduce the new time scale ς and let $'$ denote $\frac{d}{d\varsigma} = r\frac{d}{dt}$ where $r = |Q|$.

Then

$$Q' = rP^T \qquad H = \tfrac{1}{2}|P|^2 - \tfrac{1}{r} = h < 0$$
$$P' = -\frac{Q^T}{r^2}$$

Starting from these and the equation $t' = r$ relating the two timescales, one finds

$$t' = r \qquad\qquad Q' = rP^T$$
$$t'' = r' = PQ \qquad\qquad Q'' = r'P^T - \frac{Q}{r}$$
$$t''' = r'' = 1 + 2rh \ \ (\text{using } H = h) \qquad Q''' = (r''-1)P^T = 2hrP^T = 2hQ'$$
$$t'''' = 2hr' = 2ht'' \qquad\qquad Q'''' = 2hQ''$$

Now let $X = (t'', \sqrt{a}\,Q'') = \left(PQ, \sqrt{a}\left((PQ)P^T - \frac{Q}{r}\right)\right) \in \mathbb{R}^4$

and $Y = X' = \left(1 - \frac{r}{a}, -rP^T/\sqrt{a}\right) \in \mathbb{R}^4$

where $a = \frac{1}{2|h|}$. Then one can check that

$$X' = Y \qquad |X| = \sqrt{a} \qquad X^TY = 0$$
$$Y' = -\tfrac{1}{a}X \qquad |Y| = 1$$

These are the equations for the geodesic flow on the three-sphere

$$S^3_{\sqrt{a}} = \{|X| = \sqrt{a}\} \subset \mathbb{R}^4 \ !!$$

We have constructed an embedding taking the five-dimensional fixed energy manifold $\{(Q,P) \in T^*(\mathbb{R}^3 \setminus 0) : H = h = -\frac{1}{2a}\}$ into the five-dimensional unit tangent bundle $T_1 S^3_{\sqrt{a}}$. The inverse can be written

$$Q = -\sqrt{a}\left(X_0 \hat{Y} + (1-Y_0)\hat{X} \right)$$

$$P = -\frac{\hat{Y}^T}{\sqrt{a}(1-Y_0)}$$

where $X, Y \in \mathbb{R}^4$ have been split as

$$X = (X_0, \hat{X}) \qquad \hat{X}, \hat{Y} \in \mathbb{R}^3.$$
$$Y = (Y_0, \hat{Y})$$

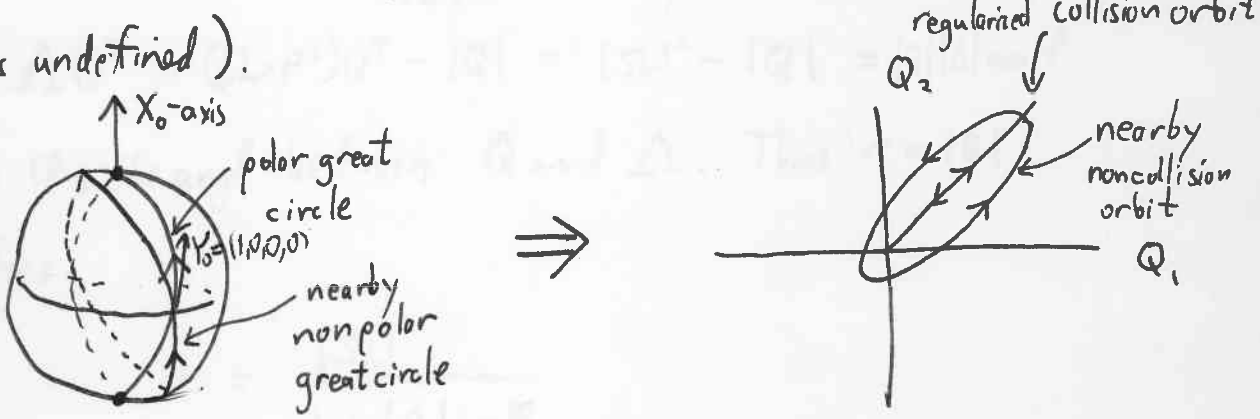
This is well-defined as long as the unit vector $Y \neq (1,0,0,0)$. Note that as $Y \to (1,0,0,0)$, we have

$$Q \to 0 \qquad |P| \to \infty$$

which corresponds to the collision singularity.

This embedding neatly regularizes the collision singularity of the Kepler problem. The nonsingular Kepler orbits of energy $h = -\frac{1}{2a}$ map to great circles in $S^3_{\sqrt{a}}$ which do not pass through the "poles" on the X_0-axis. On these orbits, the embedding is invertible. The singular Kepler orbits map to great circles which do pass through these poles. In the geodesic flow, these are nonsingular orbits so they can be followed for all time. We take the preimages to find the "regularized" orbits in $T^*(\mathbb{R}^3 \setminus 0)$.

Because the geodesic flow behaves smoothly with respect to initial conditions, this regularization extends the collision orbits smoothly with respect to initial conditions (except at the actual moment of collision when $P(t)$ is undefined).



This change of coordinates also reveals a hidden symmetry of the Kepler problem. Whereas the Kepler problem is obviously $SO(3)$ invariant, the geodesic flow on $S^3_{\sqrt{a}}$ is obviously $SO(4)$ invariant. Using the inverse embedding we obtain a not so obvious $SO(4)$ symmetry for the Kepler problem. Since $SO(4)$ is 6-dimensional (versus 3 dimensions for $SO(3)$) we can use Noether's theorem to produce 6 "angular momenta." The quantity

$$F(X,Y) = Y_0 \hat{X} - X_0 \hat{Y} \in \mathbb{R}^3$$

is a vector-valued integral leading to a new integral

$$F(Q,P) = \sqrt{a}\left[\frac{Q}{r} - \frac{Q}{a} - (PQ)P^T\right] = \sqrt{a}\left[\Omega \times P^T - \frac{Q}{|Q|}\right] \in \mathbb{R}^3$$

for the Kepler problem. This is the Laplace vector.

This extra integral can be used to explain the frequency locking on the tori $S_{(\omega_0,h)}$ and to find the orbits' shapes as well. Let

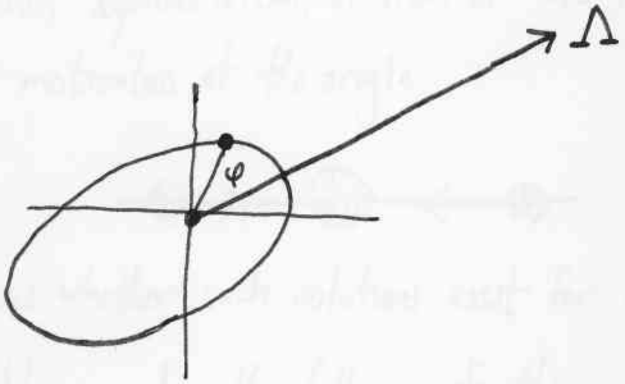
$$\Lambda = \Omega \times P^T - \frac{Q}{|Q|} \quad \text{the Laplace vector}$$

Then $\Lambda Q^T = (\Omega \times P^T) Q^T - |Q| = |\Omega|^2 - |Q| = |Q||\Lambda| \cos \varphi$

where φ is the angle between Q and Λ. Thus $r = |Q|$ satisfies

$$r = \frac{|\Omega|^2}{1 + |\Lambda| \cos \varphi}$$

which is the familiar conic section.



This forces $\Delta \varphi = \Delta \Theta = 2\pi$ and makes all orbits periodic.

Special Solutions of the n-body problem The reduction of the 2-body problem is one of the few success stories in celestial mechanics. For $n \geq 3$ one cannot give a complete description of the flow. Instead one tries to prove existence or non-existence of orbits with prescribed behavior. We will consider the simplest possible kinds of solutions first.

Euler was able to generalize the zero angular momentum collision solutions of the Kepler problem to $n=3$. He sought solutions of the collinear 3-body problem which collapse homothetically to triple collision, that is, the size of the configuration tends to 0 while the shape remains the same. For example, if two masses are equal then there is an obvious symmetrical solution of this type with the third mass remaining motionless at the origin.

It is not obvious whether such solutions exist for non equal masses.

We will consider the following further generalization. Consider the n-body problem and let $q_0 \in \mathbb{R}^{3n} \setminus \Delta$. Call q_0 a central configuration if there is some function $r(t) > 0$ such that

$$q(t) = r(t) q_0$$

is a solution to the n-body problem. Thus q_0 represents the constant shape of the configuration and $r(t)$ the size.

Substituting into Newtons equation gives

$$\ddot{r} M q_0 = \nabla U(r q_0) = r^{-2} \nabla U(q_0).$$

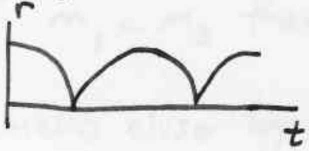
using homogeneity of the potential. Multiplying both sides by q_0^T and using homogeneity again gives

$$\ddot{r} = -\frac{\lambda}{r^2} \qquad \lambda = \frac{U(q_0)}{q_0^T M q_0}$$

as the equation of motion for the size and

$$M^{-1} \nabla U(q_0) + \lambda q_0 = 0 \qquad *$$

for the constant shape. The differential equation is the reduced zero-angular momentum Kepler problem whose solutions are represented by Newton's cycloid:



The equation * is exactly the same as the equation for critical configurations which we derived when studying integral manifolds (however, a critical configuration had to be planar.)

Thus we can construct solutions of the n-body problem by first solving *, then multiplying by any solution of the collinear Kepler problem. As noted already, * is not so easy to solve. Once again, discussion of this equation will be postponed, but to get some idea of the complexity we will look at Euler's case.

Let $x = q_2 - q_1$, $y = q_3 - q_1$ and assume the particles are in the order $q_1 < q_2 < q_3$ in $\mathbb{R}$.

Equation $*$ reduces to

$$\lambda x - \frac{(m_1+m_2)}{x^2} + \frac{m_3}{y^2} - \frac{m_3}{(x+y)^2} = 0$$

$$\lambda y - \frac{(m_2+m_3)}{y^2} + \frac{m_1}{x^2} - \frac{m_1}{(x+y)^2} = 0$$

Eliminating λ gives a single homogeneous equation for (x, y). Introducing the ratio $z = y/x$ one finds

$$f(z) = (m_1+m_2)z^5 + (3m_1+2m_2)z^4 + (3m_1+m_2)z^3 - (3m_3+m_2)z^2 - (3m_3+2m_2)z - (m_3+m_2) = 0$$

A positive root of this equation determines a collinear 3-body central configuration. It is an exercise to show that this equation always has exactly one positive real root $\bar{z}$. If $m_1 < m_3$ then $\bar{z} > 1$ while for $m_1 > m_3$, $\bar{z} < 1$ (smaller masses closer together)

$m_1 < m_3$

$m_1 > m_3$

Given any $z > 0$ there is a plane in mass space consisting of triples (m_1, m_2, m_3) for which the Eulerian central configuration has shape given by z.

These were the first explicit solutions of the three-body problem.

A similar problem was taken up by Lagrange who sought solutions of the form

$$q(t) = g(t)\cdot q_0 \qquad g(t) \in Euc(3).$$

Such a solution would have constant shape and size, changing configuration by a rigid motion. Assuming that the center of mass remains at the origin we are reduced to the case

$$q(t) = A(t)\cdot q_0 \qquad A(t) \in SO(3), \quad A\cdot q_0 = \begin{bmatrix} Aq_{01} \\ \vdots \\ Aq_{0n} \end{bmatrix}$$

A configuration $q_0 \in \mathbb{R}^{3n}\setminus\Delta$ which admits a solution of this form is called a relative equilibrium of the n-body problem.

Substituting into Newton's equation gives

$$\ddot{A}\cdot M q_0 = A\cdot \nabla U(q_0)$$

or

$$A^{-1}\ddot{A}\cdot M q_0 = \nabla U(q_0)$$

Consider the antisymmetric matrix $W(t) = A^{-1}\dot{A} \in so(3)$. As usual, W can be viewed as a cross product with an angular velocity vector $w(t) \in \mathbb{R}^3$. Now

$$\dot{W} = A^{-1}\ddot{A} - (A^{-1}\dot{A}A^{-1})\dot{A}$$

so

$$A^{-1}\ddot{A} = \dot{W} + W^2$$

$$= \begin{bmatrix} -(w_2^2+w_3^2) & -\dot{w}_3 + w_1w_2 & \dot{w}_2 + w_1w_3 \\ \dot{w}_3 + w_1w_2 & -(w_1^2+w_3^2) & -\dot{w}_1 + w_2w_3 \\ -\dot{w}_2 + w_1w_3 & \dot{w}_1 + w_2w_3 & -(w_1^2+w_2)^2 \end{bmatrix}$$

It will be shown that $w(t)$ is a constant vector. In other words the rotation $A(t)$ is a uniform rotation around a fixed axis. First we will show that $A^{-1}\ddot{A}$ is a constant matrix. Then since $A^{-1}\ddot{A} = \dot{W} + W^2$ is the decomposition into antisymmetric and symmetric parts, it follows that $W^2(t)$ is constant. From this one finds easily that $w(t)$ is constant.

We have

$$A^{-1}\ddot{A}\, q_{0i} = m_i^{-1} \nabla_i U(q_0) = \text{const} \qquad i = 1, \cdots, n$$

There are 3 cases to consider according to whether the vectors $q_{01}, \cdots, q_{0n}$ span $\mathbb{R}^3$, a two-dimensional subspace ($\mathbb{R}^2$, say), or a one-dimensional subspace ($\mathbb{R}^1$, say). In the first case, it is immediate that $A^{-1}\ddot{A}$ is constant since it is linear and constant on some basis of $\mathbb{R}^3$. Next suppose the q_{0i} span $\mathbb{R}^2$. Then the 3x3 matrix $A^{-1}\ddot{A}$ is constant on $\mathbb{R}^2$ and maps $\mathbb{R}^2$ into itself. Hence it takes the form

$$A^{-1}\ddot{A} = \begin{bmatrix} c_{11} & c_{12} & * \\ c_{21} & c_{22} & * \\ 0 & 0 & * \end{bmatrix}$$

Using the formula on the previous page this gives

$$\left.\begin{matrix} w_2^2 + w_3^2 = -c_{11} & -\dot{w}_3 + w_1 w_2 = c_{12} \\ w_1^2 + w_3^2 = -c_{22} & \dot{w}_3 + w_1 w_2 = c_{21} \end{matrix}\right] \Rightarrow \begin{matrix} w_1^2 - w_2^2 = +c_{11} - c_{22} \\ w_1 w_2 = c_{12} + c_{21} \end{matrix}$$

$$-\dot{w}_2 + w_1 w_3 = \dot{w}_1 + w_2 w_3 = 0.$$

Hence w_1 and w_2 are constant and $w_1 w_3 = w_2 w_3 = 0$.

It follows that

$$A^{-1}\ddot{A} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{bmatrix} = \text{const} \quad \text{as required.}$$

Finally, if the $q_{0i} \in \mathbb{R}$ are collinear one may assume $w(0) = \begin{bmatrix} w_{01} \\ 0 \\ w_{03} \end{bmatrix}$. Then the initial velocities $\dot{q}_i(0) = w(0) \times q_{0i}$ all lie in the plane $\mathbb{R}^2$, so the motion remains in $\mathbb{R}^2$ for all time. It follows that $w(t) = (0, 0, w_3(t))^*$ and so

$$A^{-1}\ddot{A} = \begin{bmatrix} -w_3^2 & -\dot{w}_3 & 0 \\ \dot{w}_3 & -w_3^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

* or rather, we may as well assume this for there will be some solution of this form with the same $q(t)$.

Since this must map $\mathbb{R}^1$ to itself we have $\dot{w}_3 = 0$, $w_3 = \text{const}$ as required.

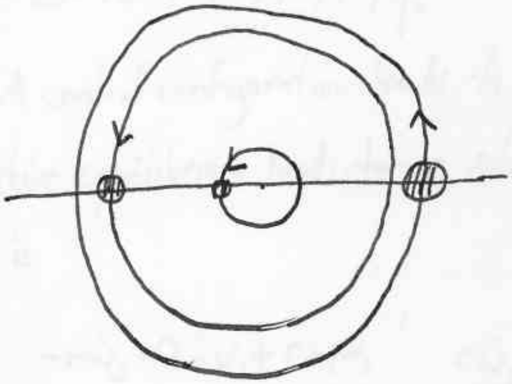
Now that we know $w(t) = w_0 = \text{const}$ we may assume without loss of generality that $w_0 = (0, 0, k)$. Then in all cases,

$$A^{-1}\ddot{A} = \begin{bmatrix} -k^2 & 0 & 0 \\ 0 & -k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

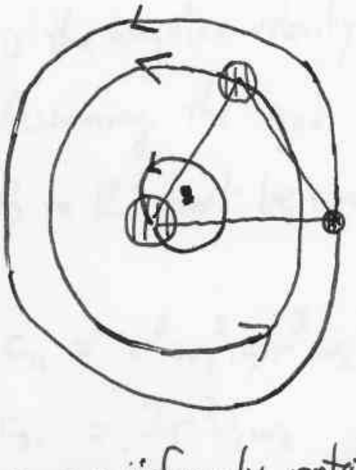
From $A^{-1}\ddot{A}\, q_{0i} = m_i^{-1} \nabla_i U(q)$ it follows that the mutual accelerations are all planar and so in fact $q_{0i} \in \mathbb{R}^2$; $i = 1, \cdots, n$. Thus the first case considered above is impossible. Given that $q_{0i} \in \mathbb{R}^2$ we can replace $A^{-1}\ddot{A}$ by $-k^2 I$ to find that q_0 satisfies:

$$M^{-1} \nabla U(q_0) + \lambda q_0 = 0 \qquad \lambda = k^2 > 0. \quad \overset{\text{again}}{\star}.$$

Thus a relative equilibrium is a planar solution of $*$, just like a critical configuration. Given a relative equilibrium we get a periodic orbit of the n-body problem which rotates rigidly about the center of mass with period $\frac{2\pi}{\sqrt{\lambda}}$. For example, the collinear central configurations of Euler are certainly planar, so there are periodic orbits of the form:



Lagrange found that the equilateral triangle is a relative equilibrium of the three-body problem for all choices of the masses (this is easy for equal masses.) We will prove this later. For now we just take note of Lagrange's periodic orbits:



One could introduce a uniformly rotating coordinate system in which such orbits appear fixed. This is the reason for the term "relative equilibrium" (it is easy to see that the n-body problem has no real equilibria!)

Combining the ideas of Euler and Lagrange one could look for more general "homographic" solutions

$$q(t) = r(t)\,A(t)\,q_0 \qquad r(t) > 0\,,\ A(t) \in SO(3).$$

Substitution into Newton's equation gives

$$r^2\left(\ddot{r}\,I + 2\dot{r}\,W + r\,A^{-1}\ddot{A}\right)q_0 = M^{-1}\nabla U(q_0)$$

where $W = A^{-1}\dot{A}$. A central configuration leads to a solution with $A(t) = I$, $W(t) = 0$ while a relative equilibrium leads to a solution with $r(t) = 1$.

The matrix on the left is

$$B(t) = r^2\begin{bmatrix} \ddot{r} - r(w_2^2 + w_3^2) & -r\dot{w}_3 - 2\dot{r}w_3 + rw_1w_2 & r\dot{w}_2 + 2\dot{r}w_2 + rw_1w_3 \\ r\dot{w}_3 + 2rw_3 + rw_1w_2 & \ddot{r} - r(w_1^2 + w_3^2) & -r\dot{w}_1 - 2\dot{r}w_1 + rw_2w_3 \\ -r\dot{w}_2 - 2\dot{r}w_2 + rw_1w_3 & r\dot{w}_1 + 2\dot{r}w_1 + rw_2w_3 & \ddot{r} - r(w_1^2 + w_2^2) \end{bmatrix}$$

where $w(t) \in \mathbb{R}^3$ is the angular velocity vector associated to the antisymmetric matrix $W(t)$. Assuming the q_{0i} span $\mathbb{R}^2$ or $\mathbb{R}^3$ (non-collinear case) the restriction of B to $\mathbb{R}^2$ must be constant so $B = \begin{bmatrix} c_{11} & c_{12} & * \\ c_{21} & c_{22} & * \\ * & * & * \end{bmatrix}$

Then
$$\begin{aligned} c_{22} - c_{11} &= r^3 w_1^2 - r^3 w_2^2 \\ c_{12} + c_{21} &= 2r^3 w_1 w_2 \end{aligned} \quad \Rightarrow \quad r^{3/2}w_1 \text{ and } r^{3/2}w_2 \text{ are constant}$$

It will be shown that, in fact, $w_1(t) = w_2(t) = 0$. In the case where q_{0i} span $\mathbb{R}^3$ one can assume without loss of generality that $w_1(0) = w_2(0) = 0$ and then $r^{3/2}w_i(t) = r^{3/2}w_i(0) = 0$, $i = 1, 2$.

The planar case needs more work since the rotational freedom is already used up in putting the particles in $\mathbb{R}^2$. In this case the matrix entries $B_{31} = B_{32} = 0$ since $B : \mathbb{R}^2 \to \mathbb{R}^2$. Using $r^{3/2} w_i = \text{const}$ one can put these equations into the form

$$\begin{bmatrix} \frac{1}{2}\dot r & r w_3 \\ r w_3 & -\frac{1}{2}\dot r \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Either $w_1 = w_2 = 0$ or $\dot r = w_3 = 0$. The latter is the relative equilibrium problem considered above so again $w(t)$ is of the required form. In the collinear case, the same trick as for relative equilibria reduces us to the same form.

Now that we have $w(t) = \begin{bmatrix} 0 \\ 0 \\ w_3(t) \end{bmatrix}$ it follows that

$$A(t) = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ and $w_3(t) = \dot\theta(t)$. It follows that

$$B(t) = r^2 \begin{bmatrix} \ddot r - r\dot\theta^2 & -r\ddot\theta - 2\dot r\dot\theta & 0 \\ r\ddot\theta + 2\dot r\dot\theta & \ddot r - r\dot\theta^2 & 0 \\ 0 & 0 & \ddot r \end{bmatrix}.$$

One can use the angular momentum integral to show that the off diagonal entries are 0. Namely, for a solution of this type

$$p_i^T = m_i \dot q_i = m_i \dot r A q_{0i} + m_i r \dot A q_{0i} = m_i A(\dot r q_{0i} + r w \times q_{0i})$$

So $\Omega = \sum_{i=1}^{n} m_i (r A q_{0i}) \times A(\dot r q_{0i} + r w \times q_{0i}) = A r^2 \sum m_i \, q_{0i} \times (w \times q_{0i})$

The third component is $\Omega_3 = r^2 \dot\theta \left(\sum_{i=1}^{n} m_i (x_{0i}^2 + y_{0i}^2) \right) = I_3 r^2 \dot\theta$

Now $I_3 \neq 0$ so $r^2\dot{\theta} = c$ for some constant, c. Differentiation shows that the off-diagonal terms of $B(t)$ vanish.

Now in all three cases $B(t)q_0 = M^{-1}\nabla U(q_0)$ implies that the entry $B_{11}(t)$ is constant. But $B_{11} = B_{22}$. Thus there is a constant λ such that

$$B(t) = \begin{bmatrix} r^2\ddot{r} - r^3\dot{\theta}^2 & 0 & 0 \\ 0 & r^2\ddot{r} - r^3\dot{\theta}^2 & 0 \\ 0 & 0 & r^2\ddot{r} \end{bmatrix} = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & * \end{bmatrix}$$

The equations

$$r^2\dot{\theta} = c$$
$$\ddot{r} - r\dot{\theta}^2 = \frac{-\lambda}{r^2}$$

are exactly the polar coordinate version of the Kepler problem!

In the case where q_{0i} span $\mathbb{R}^3$ the entire matrix $B(t)$ is constant and we have

$$r^2\ddot{r} = c_{33} \qquad r^3\dot{\theta}^2 = c'$$

Since we already have $r^2\dot{\theta} = c$ the last equations shows that either $c = c' = \dot{\theta} = 0$ or $r(t) = \text{const}$. The first case is the central configuration problem discussed above. We will show the second is impossible. It implies $\ddot{r} = 0$ so the lower right entry $B_{33}(t) = 0$. Thus $B: \mathbb{R}^3 \to \mathbb{R}^2$ and it follows from $B(t)q_0 = M^{-1}\nabla U(q_0)$ that q_0 is planar, a contradiction. Thus the only homographic solutions with non-planar configurations are the homothetic collapse solutions of central configurations.

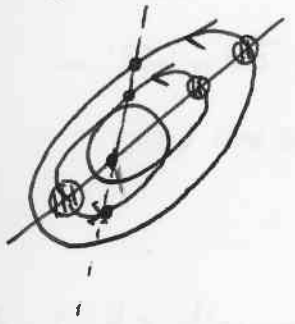
In the planar (and collinear) cases we have $B(t)q_0 = -\lambda q_0$ so, once again,

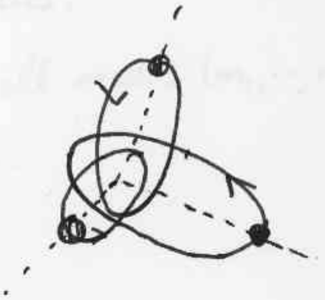
$$M^{-1}\nabla U(q_0) + \lambda q_0 = 0 \qquad *$$

Given any planar solution of $*$ and any solution $r(t), \theta(t)$ of the Kepler problem we have a solution of the n-body problem with

$$q_i(t) = r(t)\begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} q_{0i}$$

Choosing elliptical periodic solutions and using Euler and Lagrange's solutions of $*$ we get:





The configuration always maintains the same shape but changes in size and orientation as the particles move around their ellipses.

Central Configurations

We have seen that equation *

$$M^{-1}\nabla U(q) + \lambda q = 0 \qquad *$$

on $X = \mathbb{R}^{3n} \setminus \Delta$ is important for finding critical points of the integral manifolds, central configurations, relative equilibria and general homographic motions. Central configurations are more general because in the other problems we also require the configuration to be planar. So we will call * the central configuration equation. The abbreviation CC will be used for "central configuration.". The CC equation is a system of $3n$ real-analytic equations in $3n$ unknowns. It is well-understood only for $n \leq 3$. To give an idea of the lack of knowledge for higher n we note that the following are open problems:

- Find all CC of 4 equal masses.
- For $n \geq 5$, show that for almost all masses $(m_1, \cdots, m_n)$ there are only finitely many CC.

In spite of this discouraging ignorance, there are many interesting theorems about central configurations (and lots of room for further work).

One of the most fruitful lines of attack is to give the CC equation a variational interpretation. To describe this, introduce a metric on $\mathbb{R}^{3n}$ by using the mass matrix:

$$\langle q, q \rangle = q^T M q$$

We will restrict attention to the "sphere"

$$S = \{ q \in \mathbb{R}^{3n} \setminus \Delta : \langle q, q \rangle = 1, \ m_1 q_1 + \cdots m_n q_n = 0 \}$$

Note that * has the property that if q is a solution so is kq for any $k \in \mathbb{R}$, $k \neq 0$ (the value of λ changes to $\lambda |k|^{-3}$). Thus there is no essential loss of generality if fixing the size of the configuration by setting $\langle q, q \rangle = 1$. The center of mass condition holds automatically for solutions of * since we have

$$\lambda \left(\sum_{i=1}^{n} m_i q_i \right) = -\sum_{i=1}^{n} \nabla_i U|_q) = 0.$$

Note, by the way, that λ is uniquely determined by q since

$$\lambda \left(\sum m_i |q_i|^2 \right) = -\sum_{i=1}^{n} q_i^T \nabla_i U(q_i) = +U(q)$$

$$\lambda = \frac{U(q)}{\langle q, q \rangle}$$

using the fact that U is homogeneous of degree -1. Now on S, the CC equation is

$$F(q) = M^{-1} \nabla U(q) + U(q) q = 0.$$

It turns out that this is the gradient vectorfield of the restriction $U|_S$ with respect to our metric $\langle \, , \rangle$. To show this one must check that F is tangent to S and that for $v \in T_q S$:

$$DU(q) v = \langle F, v \rangle$$

Now the normal to S with respect to $<,>$ is q and we have

$$<F, q> = <M^{-1}\nabla U, q> + U(q)<q,q>$$

$$= q^T M M^{-1} \nabla U(q) + U(q) \cdot 1$$

$$\text{homogeneity} \curvearrowright = -U(q) + U(q) = 0 \quad \text{as required.}$$

Also if $<v,q> = 0$,

$$<F,v> = <M^{-1}\nabla U(q), v> + 0$$

$$= (\nabla U(q)^T M^{-1}) M v = DU(q)\, v .$$

If we denote the gradient on S with respect to $<,>$ by $\tilde{\nabla}$ we can write $*$ as:

$$\tilde{\nabla} U|_S = 0 .$$

Thus we can view a central configuration as a critical point of a real-analytic function on S. Alternatively we can consider the gradient flow on S:

$$q' = \tilde{\nabla} U|_S$$

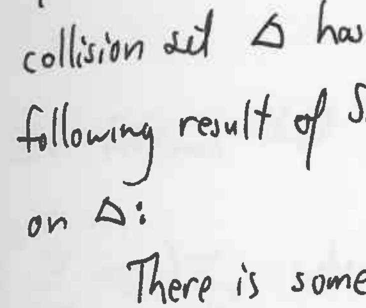
and then the central configurations are the rest points of the flow. (Note: the dynamics of this flow bear no obvious relationship to the dynamics of the n-body problem. One lives on a sphere in configuration space, the other on phase space).

If $A \in O(3)$ acts on configuration space via $A \cdot q = (Aq_1, \cdots, Aq_n)$ then A maps S to itself and is a symmetry of F: $F(Aq) = AF(q)$. It follows that if q is a CC so is Aq for all $A \in O(3)$. Thus the critical points of $U|_S$ are not isolated but rather occur as manifolds of critical points. It also follows from symmetry that given a plane or line through the origin in $\mathbb{R}^3$, the set of configurations, q, such that all of the q_i lie in the given plane or line is invariant under the gradient flow. If we want to find planar CC or collinear CC we can restrict attention to these subflows.

Note that the manifold S is non-compact. It is an open subset of a sphere of dimension $3n-4$. Since the collision set Δ has been deleted, S is not closed, so not compact. The following result of Shub shows that the CC's do not accumulate on Δ:

There is some neighborhood of Δ in S which contains no central configurations.

Otherwise there would be some $\bar{q} \in \Delta$ and a sequence of CC $q^k \to \bar{q}$. Since $\bar{q} \in \Delta$, some $\bar{q}_i$ coincide. This defines a partition of the particles into "clusters". For k large the q_i^k, q_j^k from the same cluster will be close while the q_i^k, q_j^k from different clusters will be bounded away from one another. Let $\gamma \subset \{1, \cdots, n\}$ be the indices of a cluster.



Let $F_i = \nabla_i U$ be the force on the ith particle. Since q^k is a CC we have

$$F_i(q^k) = -\lambda_k m_i q_i^k \,, \quad \lambda_k = U(q^k)$$

hence

$$\sum_{i\in\gamma} F_i = -\lambda_k \sum_{i\in\gamma} m_i q_i^k \,.$$

Since $\bar{q} \in \Delta$, $\lambda_k = U(q^k) \to \infty$ as $k \to \infty$. But the left hand side remains bounded because

$$F_i = \sum_{\substack{j\in\gamma \\ j\neq i}} F_{ij} + \sum_{l\notin\gamma} F_{il}$$

$$\sum_{i\in\gamma} F_i = \overset{0 \text{ since } F_{ij}=-F_{ji}}{\cancel{\sum_{\substack{i,j\in\gamma \\ i\neq j}} F_{ij}}} + \overset{\text{bounded by def. of cluster}}{\sum_{\substack{i\in\gamma \\ l\notin\gamma}} F_{il}}$$

It follows that $\sum_{i\in\gamma} m_i q_i^k \to 0$. But this limit is just $\left(\sum_{i\in\gamma} m_i\right)\bar{q}_\gamma$ where $\bar{q}_\gamma$ is the common value of $\bar{q}_i$ for all $i \in \gamma$. Thus $\bar{q}_\gamma = 0$. Since this must hold for all γ we have $\bar{q} = 0$. However this is a contradiction because of the normalization $\langle \bar{q}, \bar{q} \rangle = 1$.

As a corollary of this result we find that there is an a priori bound on the potential of a CC, that is, there is some $K > 0$ such that $U(q) \leq K$ for all CC, q. This bound depends on the masses, however.

Some of the interesting results about CC involve the Morse index. Recall that if q is a critical point of a smooth function, V on a manifold S, there is a well-defined quadratic form on T_qS called the Hessian, given in local coordinates by the symmetric matrix of second partial derivatives:

$$H(q)(v) = v^T D^2V(q)\, v.$$

For any curve $\gamma(t)$ in S with $\gamma(0)=q$ we have $V(\gamma(t)) = V(q) + \frac{t^2}{2} H(q)(\gamma'(0)) + O(t^3)$.

Given a quadratic form on a vector space there is an associated bilinear form (with the same matrix):

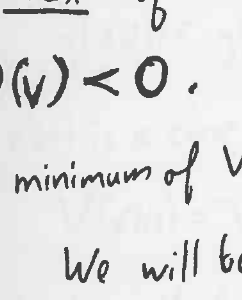
$$H(q)(v,w) = v^T D^2V(q)\, w$$

The nullity of $H(q)$ is the dimension of the null space:

$$\{ v : H(q)(v,w) = 0 \text{ for all } w \}$$

The index of $H(q)$ is the dimension of a maximal subspace on which $H(q)(v) < 0$. This is also called the Morse index of q. Thus a local minimum of V has index 0.

We will be concerned with $V: S \to \mathbb{R}$, $V(q) = U(q)$, that is, the restriction $V = U|_S$. However, we want to avoid introducing local coordinates in S. So we will try to find a quadratic form on $T_q\mathbb{R}^{3n}$ whose restriction to T_qS is the Hessian of V. The quadratic form $D^2U(q)$ is not the correct one. The problem is that S differs from T_qS to second order near q and at the same time the function $U(q)$ changes to first order (q is not a critical point of $U(q)$ in $\mathbb{R}^{3n}$, only of its restriction to S)...



Consider the function $g(q) = \sqrt{q^T M q}\,\mathcal{U}(q)$ on $\mathbb{R}^{3n}$. Clearly $g|_S = \mathcal{U}|_S = V$ but now a CC, q, is a critical point of g in $\mathbb{R}^{3n}$, not just its restriction. To see this just compute

$$Dg(q) = \sqrt{q^T M q}\; D\mathcal{U}(q) + \mathcal{U}(q)\,\frac{q^T M}{\sqrt{q^T M q}}$$

so for $q \in S$ a CC and $v \in T_q\mathbb{R}^{3n}$,

$$\begin{aligned} Dg(q)\,v &= D\mathcal{U}(q)\,v + \mathcal{U}(q)\,q^T M v \\ &= \langle M^{-1}\nabla\mathcal{U}(q) + \mathcal{U}(q)\,q, v\rangle = 0. \end{aligned}$$

It follows that for any curve $\gamma(t)$ in $\mathbb{R}^{3n}$ with $\gamma(0)=q$,

$$g(\gamma(t)) = g(q) + \frac{t^2}{2} D^2g(q)(\gamma'(0)) + O(t^3).$$

If $\gamma(t)$ is a curve in S this becomes

$$V(\gamma(t)) = V(q) + \frac{t^2}{2} D^2g(q)(\gamma'(0)) + O(t^3)$$

which shows that (with $v = \gamma'(0)$)

$$D^2g(q)(v) = H(q)(v)$$

that is $H(q) = D^2(q)\big|_{T_qS}$. Now

$$\begin{aligned} D^2g(q) &= \sqrt{q^T M q}\, D^2\mathcal{U} + \frac{1}{\sqrt{q^T M q}}\left(q^T M \otimes D\mathcal{U}(q) + D\mathcal{U}(q) \otimes q^T M\right) \\ &\quad + \frac{\mathcal{U}(q)}{\sqrt{q^T M q}}\, M - \frac{\mathcal{U}(q)}{(q^T M q)^{3/2}}\, q^T M \otimes q^T M \end{aligned}$$

For $q \in S$, $q^T M q = 1$ and for $v \in T_q S$, $q^T M v = 0$ so we find that

$H(q)$ can be represented by the symmetric $3n \times 3n$ matrix:

$$H(q) = D^2U(q) + U(q)M.$$

A straight forward computation shows that D^2U takes the form

with 3×3 blocks

$$D^2U(q) = \begin{array}{c c} \left[\begin{array}{c|c|c|c} & & & \\ \hline & D_{ii} & D_{ij} & \\ \hline & & & \\ \hline & & & \end{array}\right] & \begin{array}{c} 3 \\ 3 \\ \vdots \\ 3 \\ 3 \end{array} \\ \begin{array}{cccc} 3 & 3 & \cdots\ 3 & 3 \end{array} & \end{array}$$

$$D_{ij} = \frac{m_i m_j}{r_{ij}^3}\left(I - 3u_{ij}u_{ij}^T\right) \quad \text{for } i \neq j$$

$$D_{ii} = -\sum_{j\neq i} D_{ij}$$

where $r_{ij} = |q_i - q_j|$ and $u_{ij} = \dfrac{(q_i - q_j)}{r_{ij}}$.

As noted already, the critical points (CC's) are not isolated because of the symmetry. Let $A(t)$ be a curve of matrices in $SO(3)$ with $A(0) = I$. Then for $q \in S$, $A(t)\cdot q \in S$ and $U(A(t)\cdot q) = U(q)$. It follows that

$$DU(A(t)q)\,A(t) = DU(q)$$

Differentiating w.r.t. t at $t=0$ gives

$$q^T W^T D^2U(q) + DU(q)W = 0 \quad , W = \dot{A}(0)$$

Since q is a CC, $DU(q) = -U(q)q^TM$ and using $W^T = -W$ gives

$$q^T W^T\left(D^2U(q) + U(q)M\right) = 0$$

This means that $v = Wq$ is in the null space of the Hessian.

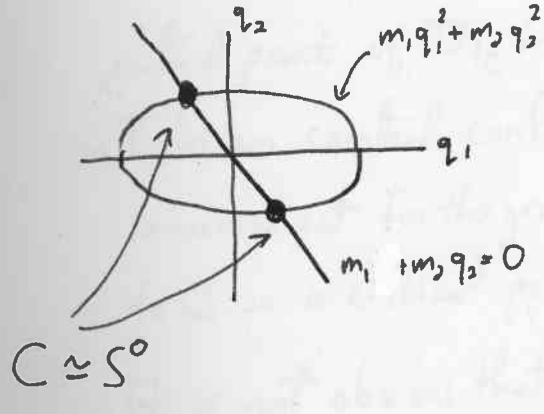
If q is a noncollinear CC then the $SO(3)$ orbit of q is 3 dimensional and this means that the vectors of the form $W \cdot q$, $W = \dot{A}(0)$, $A(t) \in SO(3)$ span a 3-dim space. Hence the nullity of $H(q)$ is at least 3. If q is collinear then it is fixed by the one parameter group of rotations around the line so there are only 2 independent vectors $W \cdot q$. Thus

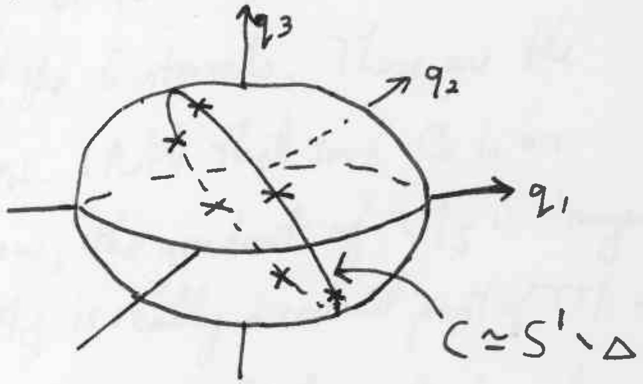
The nullity of $H(q)$ is at least 3 for noncollinear CC's and at least 2 for collinear ones.

Collinear CC's It is natural to start the search for CC's with the collinear case. Assume that $q_i \in \mathbb{R}$; $i = 1, \cdots, n$ and set

$$C = \{ q \in \mathbb{R}^n \setminus \Delta : m_1 q_1 + \cdots + q_n q_n = 0, \ m_1 q_1^2 + \cdots + m_n q_n^2 = 1 \}$$

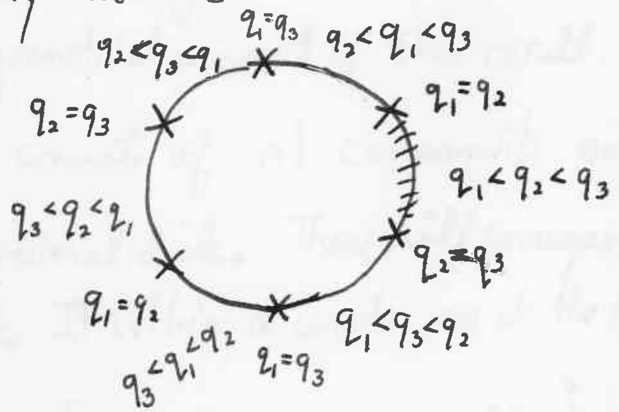
Then C is an open subset of an $(n-2)$-dimensional ellipsoid (topologically a sphere S^{n-2}). One can visualize the cases $n = 2, 3$



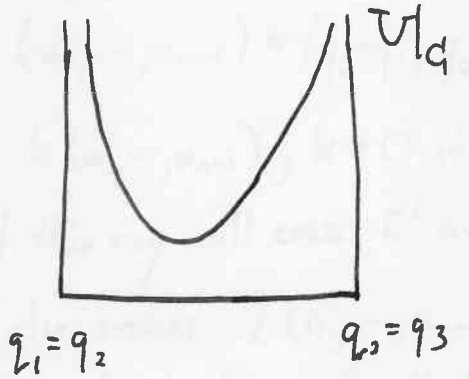


For $n=2$, every configuration is collinear along some line and every configuration is a CC (as is easily seen). For $n = 3$, C is topologically a circle with 6 points deleted (intersections with planes $q_1 = q_2, q_1 = q_3, q_2 = q_3$).

Schematically we have



There are $3! = 6$ components of C corresponding to the 6 orderings of the particles along the line. Imagine fixing one ordering, say $q_1<q_2<q_3$ (the shaded arc). Introducing some angular variable in the resulting arc of the circle, the graph of $U|_{C_1}$ must be of the form:



that is, $U|_{C_1} \to \infty$ at the boundary of the arc. Now it is obvious that there is at least one critical point of $U|_{C_1}$ in each of the 6 intervals. These are the Eulerian central configurations. (Note that since C_1 is an invariant set for the gradient flow, the gradient of $U|_S$ is tangent to C_1 so a critical point of $U|_{C_1}$ is really a critical point of $U|_S$ as well). It is not obvious that there is only one critical point in each interval. This follows from studying Euler's 5th degree equation or from the generalization discussed below.

Moulton was able to generalize Euler's CC's to the n-body problem. We will give a geometrical account of this result. First we will show that C consists of $n!$ components each homeomorphic to an $(n-2)$ dimensional disk. These will correspond to the $n!$ orderings of the points. It suffices to consider one of the orderings, so let

$$C' = \{ q \in C : q_1 < q_2 \cdots < q_n \}$$

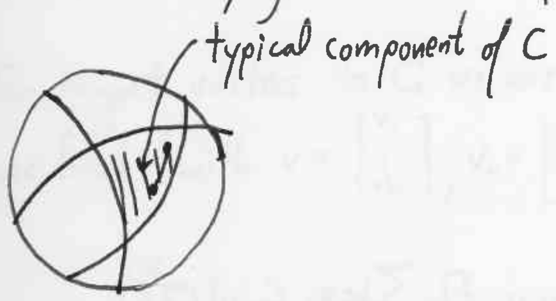
Also introduce

$$C'' = \{ q \in \mathbb{R}^n \setminus \Delta : m_1 q_1 + \cdots + m_n q_n = 0, \; q_1 < \cdots < q_n \}$$

that is, C' is obtained from C'' by normalization: $m_1 q_1^2 + \cdots + m_n q_n^2 = 1$. Clearly C'' is diffeomorphic to $(\mathbb{R}^+)^{n-1}$ by the map sending $(q_1, \cdots, q_n)$ to $(u_1, \cdots, u_{n-1}) = (q_2 - q_1, q_3 - q_2, \cdots, q_n - q_{n-1})$. Consider a ray $k(u_1, \cdots, u_{n-1})$, $k > 0$ in $(\mathbb{R}^+)^n$. It determines a ray in C'' and this ray will cross C' in a unique point. But the ray in $(\mathbb{R}^+)^{n-1}$ also crosses $\{ (u_1, \cdots, u_{n-1}, 1) \}$ in a unique point so C' is diffeomorphic to this set, that is, to $(\mathbb{R}^+)^{n-2}$ which in turn is topologically a disk. Thus the set C' and the others corresponding to different orderings are connected sets. On the other hand points of C with different orderings certainly are not in the same component of C, since to change ordering by a continuous deformation one must pass through a configuration in Δ. Thus there are exactly $n!$ components, as claimed.

The existence of a central configuration in each component now follows from the fact that $U \to \infty$ as we approach the bounday — there must be a minimum of $U|_{C_1}$ in each component.

For the uniqueness, we need a convexity argument using the Hessian. We will make use of the geodesics of the metric $\langle , \rangle$ on C. Since C is isometric to the standard round sphere in Euclidean $(n-1)$ space we will call these geodesics "great circles". (The map $x_i = \sqrt{m_i}\, q_i$ is an isometry from $(\mathbb{R}^n, \langle , \rangle)$ to $\mathbb{R}^n$ with the Euclidean metric). Since the components of C (like C' above) can be expressed as intersections of C with half-spaces of the form $\{q_i < q_j\}$ they are geodesically convex, that is, given two points q, q' in a component, there is an arc of a great circle connecting them and lying in the same component. If $\gamma(s)$ is a unit speed parametrization of a great circle of C then $\gamma''(s) = -\gamma(s)$ just as on the standard unit sphere. Now



consider the function $U(\gamma(s))$. We have

$$U(\gamma(s))' = DU(\gamma(s))\,\gamma'$$

$$U(\gamma(s))'' = D^2U(\gamma(s))(\gamma',\gamma') + DU(\gamma)\overset{-\gamma(s)}{\gamma''}$$

$$= D^2U(\gamma(s))(\gamma',\gamma') + U(\gamma(s)) \quad \text{by homogeneity}$$

We will use our formula for D^2U to show $U(\gamma(s))'' > 0$. It follows that there is at most one critical point in each component of C_1. Otherwise we could connect with a geodesic and we would have $U(\gamma(s))' = 0$ at both endpoints which forces a zero of $U(\gamma(s))''$ in between, a contradiction.

Since q is collinear the unit vectors $u_{ij} = \dfrac{q_i - q_j}{r_{ij}}$ are all $\begin{bmatrix}\pm 1\\ 0\\ 0\end{bmatrix}$ and the blocks D_{ij} of the second derivative become

$$D_{ij} = \frac{m_i m_j}{r_{ij}^3}\begin{bmatrix}-2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad D_{ii} = -\sum_{j\neq i} D_{ij}$$

It is convenient to rearrange the $3n$ coordinates of q as $(x_1,\cdots,x_n,y_1,\cdots,y_n,z_1\cdots z_n)$. Then

$$D^2U = \begin{array}{c} \begin{array}{ccc} n & n & n \end{array} \\ \left[\begin{array}{c|c|c} -2B & 0 & 0 \\ \hline 0 & B & 0 \\ \hline 0 & 0 & B \end{array}\right] \end{array} \begin{array}{c} \\ n \\ n \\ n \end{array} \qquad \text{where } B \text{ is the } n\times n \text{ matrix:}$$

$$B = \begin{array}{c} \\ \\ i \\ \\ \end{array} \left[\begin{array}{ccc} & i & j \\ & \vdots & \vdots \\ & -\sum_{j\neq i}\frac{m_i m_j}{r_{ij}^3} & \frac{m_i m_j}{r_{ij}^3} \\ & \vdots & \vdots \end{array}\right].$$

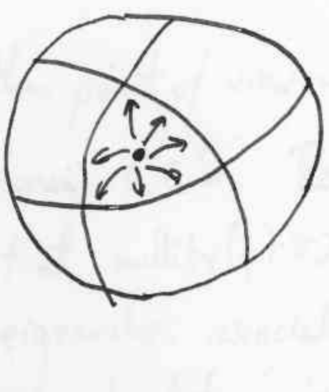
For tangent vectors to C we are concerned only with the upper left block and we find (with $v = \begin{bmatrix}v_1\\ \vdots\\ v_n\end{bmatrix}$, $v_i = \begin{bmatrix}u_i\\ 0\\ 0\end{bmatrix} \in \mathbb{R}^3$):

$$D^2U(v,v) = -2\sum_{(i,j)} B_{ij}u_iu_j = 2\overset{\text{diagonals}}{\overset{\swarrow}{\sum_{i=1}^{n}\Big(\sum_{j\neq i}\frac{m_i m_j}{r_{ij}^3}\Big)u_i^2}} - 2\overset{\text{off diagonals}}{\overset{\swarrow}{\sum_{\substack{(i,j)\\ i\neq j}}\frac{m_i m_j}{r_{ij}^3}u_i\cdot u_j}}$$

$$= 2\sum_{\substack{(i,j)\\ i<j}}\frac{m_i m_j}{r_{ij}^3}(u_i - u_j)^2 \geq 0$$

Moreover equality holds only for $(u_1,\cdots,u_n) = k(1,1,\cdots,1)$ which is not tangent to C. Applying this to $v = \gamma'(s)$ as above completes the proof of uniqueness of the collinear CC of a given ordering of masses.

It follows that the collinear CC's are nondegenerate minima of $U|_{c_1}$.
We have the following schematic picture of the gradient flow of $U|_{c_1}$:



Δ divides the (n-2) sphere into $n!$ components each containing a single restpoint of the gradient flow. Other solution flow away from the restpoint in direction of increasing $U|_{c_1}$ toward Δ where $U \to \infty$.

Before analyzing the collinear CC's further we will make a digression on the relationship between the Hessian $H(q)$ at a critical point and the linearization $D\tilde{\nabla}V(q)$ of the gradient flow at the same point.
$H(q)$ is a quadratic form which is well-defined independent of any metric.
$D\tilde{\nabla}V(q)$ is a linear map $T_qS \to T_qS$. They are related by

$$H(q)(v,w) = \langle v, D\tilde{\nabla}V(q)w \rangle = \langle D\tilde{\nabla}V(q)v, w \rangle$$

for $v,w \in T_qS$.

Since our metric is $\langle v,w\rangle = v^T M w$ we find the matrix of $D\tilde{\nabla}V(q)$ is

$$D\tilde{\nabla}V(q) = M^{-1}D^2U(q) + U(q)I = M^{-1}(H(q))$$

The formula relating H & $D\tilde{\nabla}V$ shows that $D\tilde{\nabla}V$ is a symmetric linear map with respect to $\langle , \rangle$. It follows that it can be diagonalized in some $\langle , \rangle$-orthonormal basis. In such a basis it is easy to show

nullity(H) = dim(kernel($D\tilde{\nabla}V$)) = (multiplicity of $\lambda = 0$ as eigenvalue)
index(H) = (total multiplicity of all negative eigenvalues $\lambda < 0$)

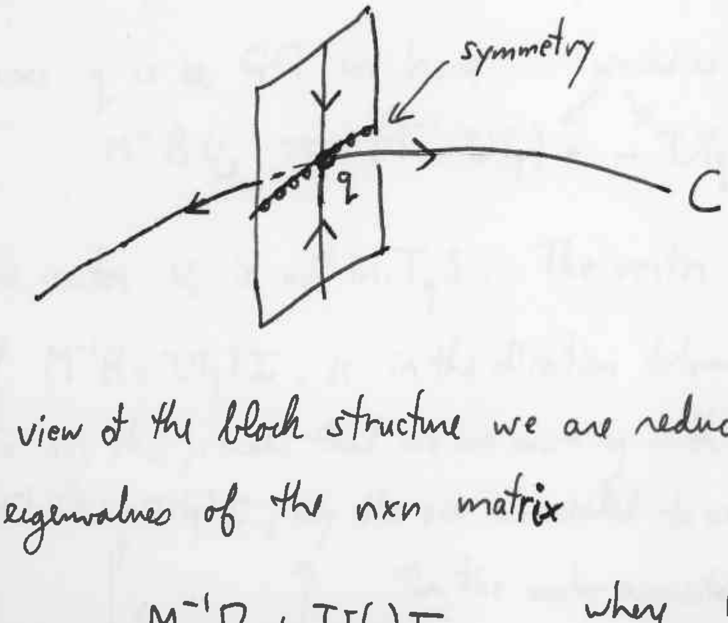
From a dynamical point of view we have

$$\text{Morse index of } q = \text{dimension of } W^s(q) \quad \text{(stable manifold for grad. flow)}$$

With this point of view we will turn to the computation of the Morse index of the collinear CC's. Recall that the dimension of S is $(3n-4)$. We know that $\text{nullity}(q) \geq 2$ by symmetry and we have just seen that the $(n-2)$ eigenvalues associated to $T_q C_1$ are all positive. This leaves $(2n-4)$ eigenvalues to determine. It will be shown that they are all negative. Thus:

If q is a collinear CC then $\text{nullity}(q) = 2$ and $\text{index}(q) = 2n-4$.

In particular there is no degeneracy beyond that due to symmetry. It is natural to call a CC <u>nondegenerate</u> in this case. Dynamically we have stability in the $(2n-4)$ directions corresponding to the negative eigenvalues and we have the following schematic of the gradient flow near q:



In view of the block structure we are reduced to the problem of finding the eigenvalues of the $n \times n$ matrix

$$M^{-1}B + U(q)I \qquad \text{where } M = \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{bmatrix}, \ I \ n\times n \text{ identity}$$

Recall that finding the coordinates of q explicitly is out of the question (recall Euler's 5th degree equation). The following interesting proof is due to Conley.

Consider the linear differential equation

$$\dot{u} = M^{-1}Bu \,, \qquad M^{-1}B = \begin{array}{c} \\ i \\ \\ \end{array}\left[\begin{array}{cccc} & i & j & \\ & \vdots & \vdots & \\ \cdots & -\sum_{j\neq i}\frac{m_j}{r_{ij}^3} & \frac{m_j}{r_{ij}^3} & \cdots \\ & \vdots & \vdots & \end{array}\right]$$

There are 2 simple eigenvalues

$\lambda = 0$ with eigenvector $u_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$\lambda = -U(q)$ with eigenvector $u_2 = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ where $q_i = \begin{bmatrix} x_i \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3$.

To see the latter note that the ith component of $M^{-1}Bu_2$ is

$$(M^{-1}Bu_2)_i = -\Big(\sum_{j\neq i}\frac{m_j}{r_{ij}^3}\Big)x_i + \sum_{j\neq i}\frac{m_j x_j}{r_{ij}^3} = \sum_{j\neq i}\frac{m_j(x_j - x_i)}{r_{ij}^3}$$

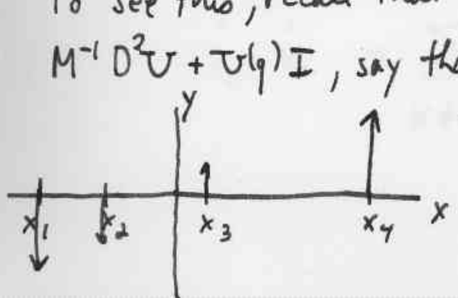
$$= m_i^{-1}\nabla_i U(q)$$

Since q is a CC we have

$$M^{-1}Bu_2 = M^{-1}\nabla U(q) = -U(q)q = -U(q)u \quad \text{as required.}$$

viewed as being in $\mathbb{R}^n$

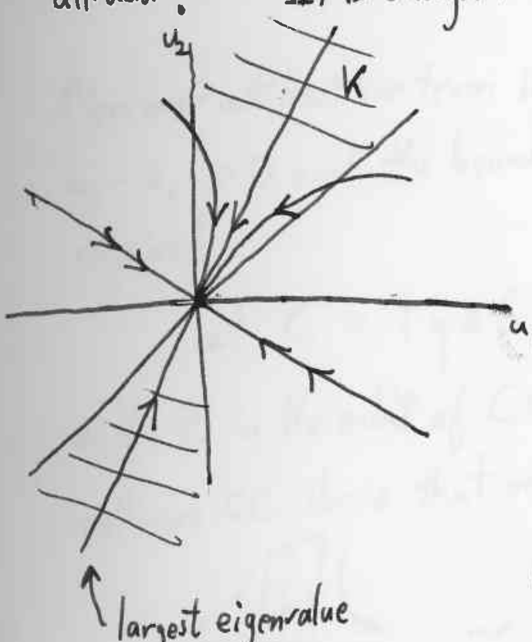
The vector u_1 is not in T_qS. The vector u_2, which is in the kernel of $M^{-1}B + U(q)I$, is in the direction determined by the rotational symmetry. To see this, recall that we are working with one of the lower 2 blocks of $M^{-1}D^2U + U(q)I$, say the one associated to vectors of the form $v = \begin{bmatrix} v_1 \\ \vdots \end{bmatrix}$, $v_i = \begin{bmatrix} 0 \\ u_i \\ 0 \end{bmatrix}$



Then the vector associated to rotation of q in the (x,y) plane is exactly $v_i = \begin{bmatrix} 0 \\ x_i \\ 0 \end{bmatrix}$, that is, u_2.

Showing that the other (n-2) eigenvalues of $M^{-1}B + U(q)I$ are negative is equivalent to showing that 0 and $-U(q)$ are the largest eigenvalues of $M^{-1}B$ or that $-U(q)$ is the largest when we restrict to vectors orthogonal to u_1.

Conley observed that this is equivalent to showing that in the flow on the space of lines through the origin determined by $\dot{u} = M^{-1}Bu$, the line determined by u_2 is an attractor. It is enough to find a "cone", K, around this line which is carried strictly inside itself by the flow (except for the origin).



Let q be the collinear CC with ordering $x_1 < x_2 < \cdots < x_n$. The let

$$K = \{u : m_1u_1 + \cdots + m_nu_n = 0,\ u_1 \le u_2 \cdots \le u_n\}$$

Note that the boundary ∂K consists of points where one or more equalities hold. However, except for the origin, at least one inequality must hold (otherwise $u = k(1,1,\cdots,1)$). Consider a boundary point with

$$u_{i-1} \le u_i \le \cdots \le u_j \le u_{j+1} \qquad , j > i$$

The differential equation is

$$\dot{u}_i = \sum_{k \ne i} \frac{m_k}{r_{ik}^3}(u_k - u_i) \qquad \dot{u}_j = \sum_{k \ne j} \frac{m_k}{r_{jk}^3}(u_k - u_j)$$

Since $u_i = u_j$ we get

$$\dot{u}_j - \dot{u}_i = \sum_{k \ne i j} m_k (u_k - \overset{\text{or } u_j}{u_i}) \left[\frac{1}{r_{jk}^3} - \frac{1}{r_{ik}^3} \right]$$

Every term in this sum is non-negative since

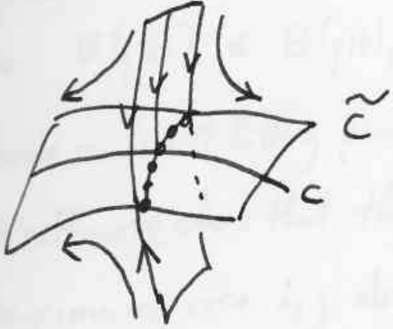
if $k < i$ $\quad (u_k - u_i) \le 0 \quad \frac{1}{r_{jk}^3} - \frac{1}{r_{ik}^3} < 0$

if $i \le k \le j$ $\quad (u_k - u_i) = 0$

if $k > j$ $\quad (u_k - u_i) \ge 0 \quad \frac{1}{r_{jk}^3} - \frac{1}{r_{ik}^3} > 0$

Moreover, at least one term is strictly positive since not all u_i are equal. Thus $\ddot{u}_j - \ddot{u}_i > 0$ and the boundary point moves into the interior of the cone as required.

Let $\tilde{C} = \{q \in S : q$ is collinear along some line$\}$. Then $C \subset \tilde{C}$ and $\tilde{C}$ is the orbit of C under $SO(3)$. The local discussion near a collinear CC shows that orbits near $\tilde{C}$ are attracted to $\tilde{C}$ by the gradient flow.



We will "globalize" this result by finding an explicit neighborhood of $\tilde{C}$ in S such that orbits of the gradient flow in the neighborhood get more and more collinear.

We will introduce a function which measures the approximate collinearity of a configuration, q. Fix $q \in S$ and a line L in $\mathbb{R}^3$ and define

$$\Theta(q, L) = \max_{i \neq j} \angle(L, q_i - q_j)$$

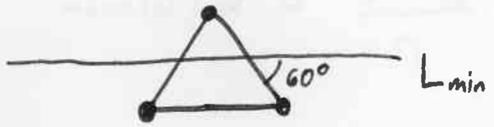
where $\angle$ denotes the angle.

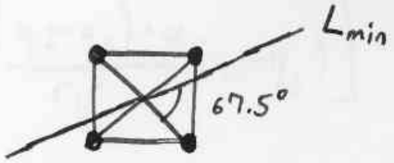
This vanishes iff q is collinear along a line parallel to L. Now let

$$\Theta(q) = \min_{L} \Theta(q, L)$$

which vanishes iff q is collinear along some line.

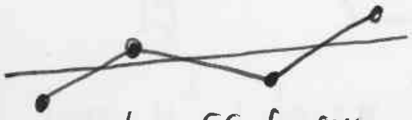
For example





We will show that in $\mathcal{U} = \{q : \Theta(q) \le 45°\}$, $\Theta(q)$ is strictly decreasing along orbits of the gradient flow $\dot q = \tilde\nabla V$. In other words if q is close to collinear (as measured by $\Theta(q)$) then it gets even more collinear under the flow. As a corollary we get the "45° theorem":

There are no central configurations with $0 < \Theta(q) \le 45°$



not a CC for any choice of masses

To prove that $\Theta(q(t)) < \Theta(q(0))$ for $t>0$ choose a "best fitting line" L with

$$\Theta(q(0)) = \Theta(q(0), L).$$

Now $\Theta(q(t)) \le \Theta(q(t), L)$ by definition, so it is enough to show $\Theta(q(t), L) < \Theta(q(0), L)$.

Choose a "worst fitting pair" of indices i,j such that $\Theta(q(0), L) = \angle(q_i - q_j, L)$. It suffices to show that the angle decreases for each such pair for then the maximum over i,j also decreases. Let $\alpha = \angle(q_i - q_j, L)$. We will show $\dot\alpha < 0$.

Now for the gradient flow,

$$\dot q_i = m_i^{-1}\nabla_i U(q) + U(q) q_i$$

and

$$\dot q_i - \dot q_j = m_i^{-1}\nabla_i U(q) - m_j^{-1}\nabla_j U(q) + U(q_i - q_j).$$

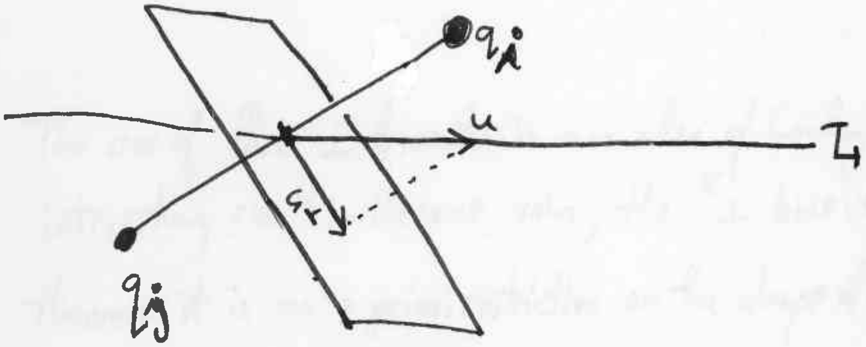
Let u be a unit vector along L. Then $\cos(\alpha(t)) = \frac{q_i - q_j}{|q_i - q_j|} \cdot u$

(assume u is chosen so this is non-negative).

This gives

$$-\sin(\alpha)\,\dot{\alpha} = \frac{(\dot{q}_i-\dot{q}_j)}{r_{ij}}\cdot\left[u - \frac{(q_i-q_j)\cdot u}{r_{ij}^2}(q_i-q_j)\right]$$

$$= \frac{(\dot{q}_i-\dot{q}_j)}{r_{ij}}\cdot u^\perp$$ where $u^\perp$ is the projection of u onto the ⊥ bisector of q_i & q_j

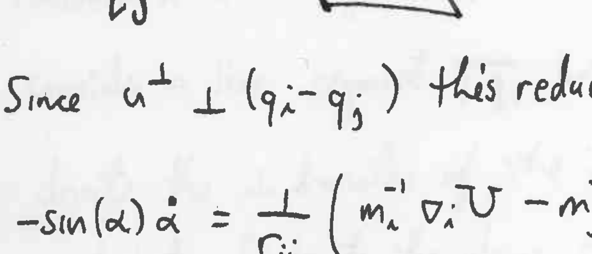


Since $u^\perp \perp (q_i-q_j)$ this reduces to

$$-\sin(\alpha)\,\dot{\alpha} = \frac{1}{r_{ij}}\left(m_i^{-1}\nabla_i U - m_j^{-1}\nabla_j U\right)\cdot u^\perp$$

$$= \frac{1}{r_{ij}}\sum_{k\neq i,j} m_k\left[\frac{(q_k-q_i)\cdot u^\perp}{r_{ik}^3} - \frac{(q_k-q_j)\cdot u^\perp}{r_{jk}^3}\right]$$

Now the masses q_k, $k\neq i,j$, must lie in the cones of angle $\theta(q)$ at q_i and q_j in the direction of L. The interection of these cones is the segment $\overline{q_iq_j}$ and the outer half-cones. Since $\theta \leq 45°$ these half-cones are on opposite sides of the ⊥ bisector



We will show that every term in the sum is non-negative and some are positive. Note that $(q_k-q_i)\cdot u_\perp = (q_k-q_j)\cdot u_\perp$ is ≥ 0 in the half cone at q_i and ≤ 0 in the half-cone at q_j. On the other hand,

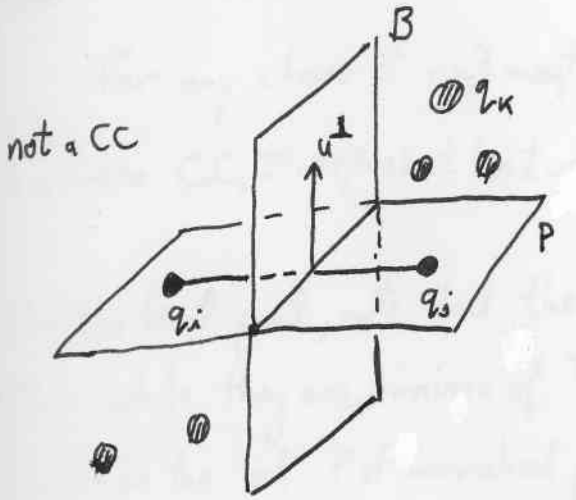
$\frac{1}{r_{ik}^3} - \frac{1}{r_{jk}^3}$ is also $\geq$ in the half cone at q_i and ≤ 0 in the half-cone at q_j

If q_k is on the line determined by q_i and q_j then the dot product is 0.

Thus every term is nonnegative. Now since $\Theta(q) > 0$, not all of the q_k lie on the line determined by q_i and q_j so at least one term is positive.

It follows that $\dot{\alpha} < 0$ completing the proof.

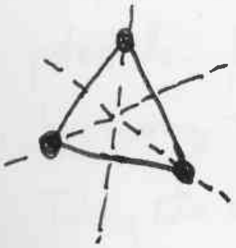
The use of the ⊥ bisector is an idea of Conley who used it to prove an interesting result in the same vein, the "⊥ bisector theorem". Like the 45° theorem it is an a priori restriction on the shape of a CC. To formulate it consider a line segment $\overline{q_i q_j}$ between two of the n particles. Let B denote the ⊥ bisector of the segment. Choose any unit vector $u^{\perp} \in B$ and let P denote the plane containing $\overline{q_i q_j}$ and normal to $u^{\perp}$. Then B and P divide $\mathbb{R}^3$ into 4 "quadrants". The ⊥ bisector theorem states:



If the masses are all contained in two diagonally opposite quadrants of B and P and not all of them lie on B ∪ P then q is not a CC for any choice of masses.

The proof is to consider $\dot{q}_i - \dot{q}_j$ for the gradient flow. If q is a CC then $\dot{q}_i - \dot{q}_j = 0$. But the formula for $(\dot{q}_i - \dot{q}_j) \cdot u^{\perp}$ is the same as in the last proof and the same argument shows that it is strictly positive, a contradiction.

Using the ⊥ bisector theorem one can give a "formula-free" proof that the only possible non-collinear central configuration of $n=3$ masses is the equilateral triangle of Lagrange. Given any pair $i \neq j$, the remaining mass must lie on the ⊥ bisector of $\overline{q_i q_j}$; otherwise it would be in one of the four open quadrants and this together with its diagonal opposite would violate the theorem. Now an equilateral triangle is the only kind with the property that every particle lies on the ⊥ bisector of the other two. A similar argument shows that the only possible non-planar CC of $n=4$ masses is the regular tetrahedron



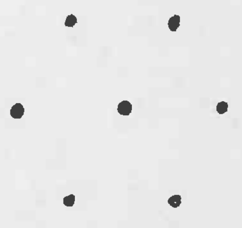
This argument does not imply that the equilateral triangle and tetrahedron actually are CC's for all masses. For this we need the following existence proof:

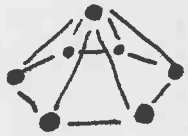
For any choice of $n \geq 3$ masses, there exists at least one planar but non-collinear CC. If $n \geq 4$, there is at least one non-planar CC.

For the first part, note that the index computation for collinear CC's shows that while they are minima of $U|_{C_1}$ they are not minima when viewed in S or in the set P of normalized planar configurations. Since $U|_P$ is bounded below (by 0) it has a minimum at some planar but not collinear CC. The second part follows in a similar way once we show that planar CC's are not minima when viewed in S.

One might hope that the set, P, of normalized planar configurations would be an attractor in S much like the set of collinear configurations. The following example shows that this is not the case. Consider a configuration of (n+1) equal masses in the form of a regular n-gon with a mass at the center.

It follows from symmetry that this is a CC in P. If P is attracting for the gradient flow then all non planar configurations nearby will have lower potentials. Define a curve of configurations in S by moving the central mass up out of the plane and the n-gon down. Taking into account the center of mass and normalization conditions we find the curve





$$q_j(s) = r(s) \begin{bmatrix} \cos\left(\frac{2\pi j}{n}\right) \\ \sin\left(\frac{2\pi j}{n}\right. \\ -\frac{s}{n} \end{bmatrix} \qquad j=1,\cdots,n$$

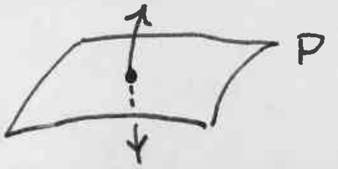
$$q_{n+1}(s) = r(s) \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \qquad r(s) = \frac{1}{\sqrt{n+\left(1+\frac{1}{n}\right)^2 t^2}}$$

where s is a parameter along the curve. Substituting into the Newtonian potential one finds

$$U(q(s))'\Big|_{s=0} \underset{\uparrow}{=} 0 \qquad U(q(s))''\Big|_{s=0} = k\left[\frac{1}{2n}\sum_{j=1}^{n}\frac{1}{\sqrt{2-2\cos\left(\frac{2\pi j}{n}\right)}} - 1\right]$$

because $q(0)$ is a CC $\qquad$ $k>0$ a const.

This last quantity should be negative if the potential of the non-planar configurations is lower. However, the sum is $O(n \log n)$ so for n sufficiently large $U''(q(s))\big|_{s=0} > 0$. In fact one can show that this is so for $n \geq 473$. Thus the planar set P is not always attracting.



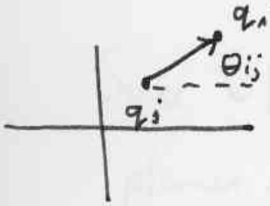
We will now see what can be said about the Morse indices of planar CC's. This will be useful both for understanding the relationship between planar and non-planar CC's and for applying the Morse inequalities later. Unfortunately, there is not much known about this question.

As above, let P denote the planar configurations in $\mathbb{R}^2$:

$$P = \{ q \in \mathbb{R}^{3n} \setminus \Delta : q_i = \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix},\ m_1 q_1 + \cdots + m_n q_n = 0,\ \langle q, q \rangle = 1 \} \subset S.$$

P is topologically an open subset of a sphere of dimension $2n-3$.

Consider the Hessian $H(q)$ at a CC, $q \in P$. The unit vectors $u_{ij} = \frac{q_i - q_j}{r_{ij}}$ in the formula take the form $\begin{bmatrix} \cos(\theta_{ij}) \\ \sin(\theta_{ij}) \\ 0 \end{bmatrix}$ where θ_{ij} is the angle between $q_i - q_j$ and the x-axis in $\mathbb{R}^2$. The 3×3 blocks of $D^2U(q)$ are



$$D_{ij} = \frac{m_i m_j}{r_{ij}^3} \begin{bmatrix} 1-3c_{ij}^2 & -3s_{ij}c_{ij} & 0 \\ -3s_{ij}c_{ij} & 1-3s_{ij}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D_{ii} = -\sum_{j \neq i} D_{ij} \qquad \begin{aligned} c_{ij} &= \cos(\theta_{ij}) \\ s_{ij} &= \sin(\theta_{ij}) \end{aligned}$$

As with the collinear case, it is convenient to reorder the $3n$ coordinates as $q = (x_1 \cdots x_n\ y_1 \cdots y_n\ z_1 \cdots z_n)$. Then

$$D^2U(q) = \begin{array}{c|c|c} & 2n & n \\ \hline 2n & D' & 0 \\ \hline n & 0 & B \end{array}$$

$$D'_{ij} = \frac{m_i m_j}{r_{ij}} \begin{bmatrix} 1-3c_{ij}^2 & -3s_{ij}c_{ij} \\ -3s_{ij}c_{ij} & 1-3s_{ij}^2 \end{bmatrix}$$

$$D'_{ii} = -\sum_{j \neq i} D_{ij}$$

$$B_{ij} = \frac{m_i m_j}{r_{ij}} \qquad B_{ii} = -\sum_{i \neq j} B_{ij}$$

$\mathbb{B}$ is just the same as in the collinear case (but much harder to estimate).
Using this splitting we can define the <u>planar index</u> and <u>normal index</u> for a planar CC and similarly a planar and normal nullity.
Namely these are the indices and nullities of

$$D' + U(q)\,I \qquad \text{and} \qquad B + U(q)\,I$$

respectively. We know from symmetry that the nullity of a planar, non-collinear CC is at least 3. Clearly, one of the 3 directions determined by the symmetry group is associated to the rotations within $\mathbb{R}^2$ and the other two take the particles out of $\mathbb{R}^2$. Hence

$$\left.\begin{array}{l}\text{planar nullity} \geq 1 \\ \text{normal nullity} \geq 2\end{array}\right. \qquad \text{for planar, non-collinear CC's.}$$

Since $U|_P$ has a minimum there exists at least one planar CC with planar index 0. On the other hand, it turns out that there is a non trivial upper bound on the planar index. The trivial upper bound would be $2n-4$ since P is $(2n-3)$ dimensional and there is at least one null direction. In fact, as Palmore proved:

$$\text{planar index} \leq n-2$$

This is just half the available dimensions and is exactly the value we found for the collinear CC's (their index <u>in S</u> was $2n-4$).

The proof uses the complex structure of $\mathbb{R}^2$, that is, the rotation by 90°. Given a vector $u \in \mathbb{R}^2$, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ let $iu = \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix}$. If $v \in T_q P$, $v = (v_1, \cdots, v_n)$, $v_i \in \mathbb{R}^2$, define $iv = (iv_1, \cdots, iv_n)$. The formula for $D^2U(q)(v)$ can be written

$$D^2U(q)(v) = \sum_{\substack{(i,j) \\ i<j}} \frac{m_i m_j}{r_{ij}^{\,3}} \left[-|v_i - v_j|^2 + 3\left(u_{ij} \cdot (v_i - v_j)\right)^2 \right]$$

and

$$D^2U(q)(iv) = \sum_{\substack{(i,j) \\ i<j}} \frac{m_i m_j}{r_{ij}^{\,3}} \left[-|v_i - v_j|^2 + 3\left(iu_{ij} \cdot (v_i - v_j)\right)^2 \right]$$

using $|iu| = |u|$ and $iu \cdot u' = u \cdot (iu')$. Now u_{ij} and iu_{ij} form an orthonormal basis for $\mathbb{R}^2$ so

$$\left(u_{ij} \cdot (v_i - v_j)\right)^2 + \left(iu_{ij} \cdot (v_i - v_j)\right)^2 = |v_i - v_j|^2.$$

Hence

$$D^2U(q)(v) + D^2U(q)(iv) = 2 \sum_{\substack{(i,j) \\ i<j}} \frac{m_i m_j}{r_{ij}^{\,3}} |v_i - v_j|^2$$

Since $H(q) = D^2U(q) + U(q)I$, we have

$$H(q)(v) + H(q)(iv) > 0 \quad \text{for all } v \neq 0.$$

Now let S_- be a maximal subspace on which $H(q)$ is negative definite. Then $S_+ = \{ iv : v \in S_- \}$ is a subspace of the same dimension on which $H(q)$ must be positive definite. It follows that

$$\text{planar index} = \dim(S_-) \leq \left[\tfrac{1}{2} \dim(T_q P) \right] = \left[\frac{2n-3}{2} \right] = n-2$$

↖ greatest integer function

Next consider the normal index of a planar CC, q. If q is collinear its normal index is $(n-2)$. If q is not collinear then of the n available normal dimensions, one is not tangent to S $(v = (1,1,\cdots,1))$ and two are null. Thus the maximum possible normal index is $(n-3)$. The example with 474 masses shows that this maximum is not always achieved.

We must estimate the eigenvalues of the linear map with matrix $M^{-1}B + U(q)I$ associated to the Hessian. Three of them are known : $U(q)$ with eigenvector $(1,1,1\cdots,1)$, and two 0's from symmetry. We will show that the sum of the other $(n-3)$ eigenvalues is negative. This amounts to showing that

$$\mathfrak{I} = -\operatorname{trace}(M^{-1}B) > (n-1)U(q)$$

It follows that

normal index ≥ 1 for a planar CC

In particular, this shows that the global minimum of $U|_S$ does not occur at a planar CC and so gives existence of at least one non-planar CC.

Now

$$\mathfrak{I} = \sum_{\substack{(i,j)\\ i\neq j}} \frac{m_j}{r_{ij}^3}$$

To compare this with $U(q)$ we will use a formula for the potential of a CC due to Pacella.

Using the notation $q_{ij} = q_i - q_j$, the definition of CC gives

$$U(q)q_i = \sum_{k \neq i} \frac{m_k q_{ik}}{r_{ik}^3} \qquad U(q)q_j = \sum_{k \neq j} \frac{m_k q_{jk}}{r_{jk}^3} \qquad i,j \in \{1,\dots,n\}.$$

Substracting these and taking the dot product with q_{ij} gives

$$U(q) = \frac{m_i + m_j}{r_{ij}^3} + \sum_{k \neq i,j} \frac{m_k}{r_{ij}^2}\left[\frac{q_{ik} \cdot q_{ij}}{r_{ik}^3} + \frac{q_{jk} \cdot q_{ji}}{r_{jk}^3}\right].$$

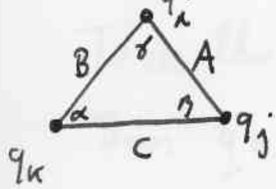
Summing these for all pairs $i \neq j$ (there are $n(n-1)$ of them) gives:

$$n(n-1)U(q) = \underset{\uparrow \atop 2I.}{\sum_{\substack{(i,j) \\ i \neq j}} \frac{m_i + m_j}{r_{ij}^3}} + \sum_{\substack{(i,j,k) \\ i \neq j \neq k \neq i}} \frac{m_k}{r_{ij}^2}\left[\frac{q_{ik} \cdot q_{ij}}{r_{ik}^3} + \frac{q_{jk} \cdot q_{ji}}{r_{jk}^3}\right]$$

so we are reduced to showing

$$(n-2)I > \sum_{\substack{(i,j,k) \\ i \neq j \neq k \neq i}} \frac{m_k}{r_{ij}^2}\left[\frac{q_{ik} \cdot q_{ij}}{r_{ik}^3} + \frac{q_{jk} \cdot q_{ji}}{r_{jk}^3}\right]$$

For each fixed k the sum runs over all triangles of masses containing q_k, counting each triangle twice. Labelling the triangles as shown, the term in the sum becomes:



$$\frac{m_k}{A^2}\left[\frac{AB\cos\gamma}{B^3} + \frac{A\cos\beta}{C^3}\right] = \frac{m_k}{A}\left[\frac{\cos\gamma}{B^2} + \frac{\cos\beta}{C^2}\right].$$

Now an elementary lemma about triangles shows $\left[\frac{\cos\gamma}{AB^2} + \frac{\cos\beta}{AC^2}\right] \leq \frac{1}{2}\left[\frac{1}{B^3} + \frac{1}{C^3}\right]$ with equality only for isoscelestriangles. This shows that the triple sum is less than

$$\sim\!\sim \leq \frac{1}{2}\sum_{\substack{(i,j,k) \\ i \neq j \neq k \neq i}} m_k\left[\frac{1}{r_{ik}^3} + \frac{1}{r_{jk}^3}\right] = \sum_{\substack{(i,j,k) \\ i \neq j \neq k}} \frac{m_k}{r_{jk}^3} = \sum_k \sum_{j \neq k} \frac{m_k}{r_{jk}^3} \cdot \sum_{i \neq j,k} 1 = (n-2)I.$$

The inequality is strict since not all of the triangles can be isosceles.

Morse Theory for Planar CC's Palmore applied Morse theory to P to estimate the number of planar CC's in the non-degenerate case (that is, assuming that the planar nullity of every CC is 1, the minimum value compatible with the symmetry). Although this is a natural assumption to make, it is very difficult to verify. It is an open problem to show that for most choices of $(m_1, \dots, m_n)$, $n \geq 5$, all planar CC's are non-degenerate (For $n = 2,3$ all CC's are known and for $n=4$ it can be shown that the exceptional masses form a proper algebraic subset of the mass space). Rather than pursue this, we will discuss the Morse inequalities.

In addition to P we will also consider its closure $\overline{P} \simeq S^{2n-3}$ and their quotient spaces under the action of the symmetry group $SO(2) \simeq S^1$. We will call the quotient spaces Q and $\overline{Q}$. Then

$$\overline{Q} \simeq CP(n-2) \quad \text{complex projective space.}$$

To see this think of the positions $q_i \in \mathbb{R}^2$ as complex numbers. Then $q \in \mathbb{C}^n$. The center of mass reduces us to $\mathbb{C}^{n-1}$

Now we form $\overline{P}$ by fixing the norm $\langle q, q \rangle = 1$. This is equivalent to identifying q with kq for all $k > 0$. Then we form $\overline{Q}$ by further identifying q and $e^{2\pi i\theta} q$ for all θ. Doing both identification at the same time we find that we are identifying q with cq for $c \in \mathbb{C}$, $c \neq 0$. This is the definition of $\mathbb{C}P(n-2)$.

Now Q is obtained from $\overline{Q}$ in the same way as P is obtained from $\overline{P}$: by deleting the collision configurations

$$P = \overline{P} \setminus \Delta \qquad Q = \overline{Q} \setminus \Delta \leftarrow \text{really the quotient of } \Delta.$$

It turns out that quotienting the $SO(2)$ action on P is simpler than for $\overline{P}$. This is because P has a global cross-section. Let

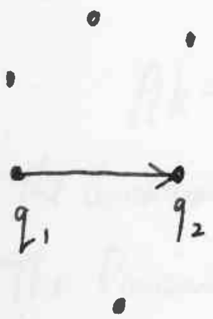
$$\Sigma = \{ q \in P : q_2 - q_1 = \begin{bmatrix} \xi \\ 0 \end{bmatrix}, \xi > 0 \}$$

These are the configurations such that the line segment $\overline{q_1 q_2}$ points along the positive x direction. Clearly there is a rotation taking any $q \in P$ to $q' \in \Sigma$ and one can use this to find a diffeomorphism

$$P \simeq \Sigma \times S^1$$

It follows that $Q \simeq \Sigma$ so

$$P \simeq Q \times S^1.$$



The simplicity of the symmetry group action on P is one reason to focus on planar configurations.

We will use the Morse inequalities in Q. Note that the one-dimensional null space of a nondegenerate CC in P disappears when we pass to Q so we can use standard Morse theory for nondegenerate critical points.

It is often difficult to do Morse theory on noncompact manifolds like Q. Recall however that we have an a priori bound on the potential of a CC from Shub's theorem. Also, the set Δ deleted from $\overline{Q}$ to get Q is just the set where $U = \infty$ so it can be written as the intersection

$$\Delta = \bigcap_{K} \{ q : U(q) \geq K \}.$$

The Morse inequalities are most easily stated in terms of Poincare polynomials. The kth Betti number of Q is

$$\beta_k = \dim (H_K (Q, \mathbb{R}))$$

the dimension of the kth homology group (with real coefficients).

The Poincaré polynomial is:

$$P_Q(t) = \sum_{k=0}^{2n-4} \beta_K t^k$$

a generating function for the Betti numbers.

The Morse polynomial of U on Q is

$$M(t) = \sum_{K=0}^{2n-4} \gamma_K t^k \qquad \gamma_K = \#\text{ of critical points of Morse index } k.$$

Then the Morse inequalities can be written:

$$M(t) = P_Q(t) + (1+t) R(t)$$

where R(t) is some polynomial with non-negative integer coefficients

We will compute the Poincaré polynomial of P instead of Q. Now $P \simeq Q \times S^1$ and the Poincaré polynomial is multiplicative (by Künneth's formula) so

$H_0(S^1) \simeq \mathbb{R}$ $\beta_0 = 1$
$H_1(S^1) \simeq \mathbb{R}$ $\beta_1 = 1$

$$P_P(t) = P_Q(t)\, P_{S^1}(t) = P_Q(t)\,(1+t)$$

To find the homology of P note that we can eliminate the center of mass condition and the normalization $\langle q, q \rangle = 1$. P can be obtained from $\mathbb{R}^{2n} \setminus \Delta$ by imposing these two conditions. The center of mass condition merely selects a cross section Σ for the two-dimensional group of translations, $\mathbb{R}^2$, and the normalization has the same effect for the group $\mathbb{R}^+$ acting as $q \to kq$, $k>0$. It follows that

$$\mathbb{R}^{2n} \setminus \Delta \simeq \mathbb{R}^2 \times \mathbb{R}^+ \times P$$

(In $\mathbb{R}^{2n}$, $\Delta = \{q : q_i = q_j \text{ for some } i \neq j\}$ contains the origin if $n>1$. If $n=1$ define $\Delta = \{0\}$)

Since the Poincaré polynomials of $\mathbb{R}^2$ and $\mathbb{R}^+$ are both 1 we may compute $P_{\mathbb{R}^{2n}\setminus\Delta}(t)$ instead of $P_P(t)$.

Now we will show inductively that

$$P_{\mathbb{R}^{2n}\setminus\Delta}(t) = (1+t)(1+2t)\cdots(1+(n-1)t)$$

For $n=1$ we have $\mathbb{R}^2 \setminus \Delta = \{q : q \neq 0\} = \mathbb{R}^2 \setminus \{0\}$. The punctured plane has the homology of a circle so $P_{\mathbb{R}^2\setminus\Delta}(t) = 1+t$.

For $n>1$ consider the map $\pi : \mathbb{R}^{2n} \setminus \Delta \to \mathbb{R}^{2(n-1)} \setminus \Delta$

$\pi(q_1, \cdots, q_n) = (q_1 \cdots q_{n-1})$. Fix $(\bar{q}_1, \cdots, \bar{q}_{n-1})$. Then

$$\pi^{-1}(\bar{q}_1, \cdots, \bar{q}_{n-1}) = \{ q = (\bar{q}_1 \cdots \bar{q}_{n-1}, q_n) : q_n \neq \bar{q}_1, \bar{q}_2 \cdots \bar{q}_{n-1} \}$$
$$\simeq \mathbb{R}^2 \text{ with } (n-1) \text{ deleted points.}$$

In fact we have a fiber bundle

$$\begin{tikzcd} F = \mathbb{R}^2 \setminus \{n-1 \text{ pts}\} \arrow[r] & \mathbb{R}^{2n} \setminus \Delta \arrow[d, "\pi"] \\ & \mathbb{R}^{2n-2} \setminus \Delta \end{tikzcd}$$

Now the Poincaré polynomial of the fiber is

$$P_F(t) = 1 + (n-1)t$$

so if this bundle were a product space we would have inductively

$$P_{\mathbb{R}^{2n}\setminus\Delta}(t) = (1+t) \cdots (1+(n-2)t) \cdot (1+(n-1)t) \quad \text{as claimed.}$$

Although it need not be a product space, it does satisfy certain topological conditions which are sufficent to show that the Poincaré polynomials do multiply. (Fund. group of base acts trivially on the fiber and there is a "section map" $\sigma : \mathbb{R}^{2n-2} \setminus \Delta \to \mathbb{R}^{2n} \setminus \Delta$ with $\pi \circ \sigma = id$).

Rather than go into the algebraic topology we will apply the result to counting CC's. The Morse inequalities for Q are

$$\sum_{k=0}^{2n-4} \gamma_k t^k = (1+2t)\cdots(1+(n-1)t) + (1+t)R(t)$$

where we have used $P_Q(t) = (1+t)^{-1} P_P(t)$. Here

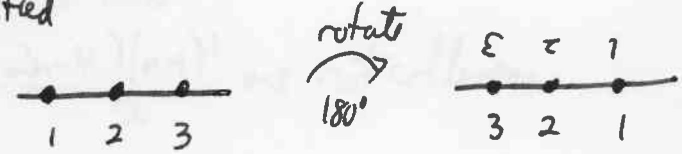
γ_K = # of CC with Morse index K.

Our upper estimate on the Morse index shows that $\gamma_K = 0,\ k \geq n-1$, so the left hand sum could be taken from 0 to (n-2). This agrees with the degree on the right.

The simplest estimate of the total number of CC's is obtained by setting $t=1$

$$\sum_{k=0}^{n-2} \gamma_K \geq 3 \cdot 4 \cdot \cdots \cdot n = \frac{n!}{2}$$

Now we know that there are n! collinear CC's. In the quotient space Q, configurations with opposite orderings are identified



so these alone account for $\frac{n!}{2}$ CC's. So far we have nothing new. However, we also know that the index of the collinear CC's is exactly n-2. We will use this in the full Morse inequalities.

$$\sum_{k=0}^{n-2} \gamma_k t^k = (1+2t)\cdots(1+(n-1)t) + (1+t)R(t).$$

The coefficients of t^{n-2} are

$$\gamma_{n-2} = (n-1)! + (r_{n-2} + r_{n-3})$$

where $R(t) = \Sigma r_k t^k$. Since $\gamma_{n-2} \geq \frac{n!}{2}$

$$r_{n+2} + r_{n+3} \geq \frac{n!}{2} - (n-1)!$$

Now setting $t=1$ gives

$$\sum_{k=0}^{n-2} \gamma_k \geq \frac{n!}{2} + 2\cdot(r_{n-2} + r_{n-3})$$

$$= \frac{3}{2}n! - 2(n-1)!$$

$$= \frac{(3n-4)(n-1)!}{2}$$

So Palmore's estimate is that there are at least

$$\frac{(3n-4)(n-1)!}{2} \text{ planar CC's}$$

of which at least

$$\frac{(2n-4)(n-1)!}{2} \text{ are not collinear}$$

(assuming non-degeneracy, of course).

n	$\frac{(2n-4)(n-1)!}{2}$	$\frac{(3n-4)(n-1)!}{2}$	
2	0	1	✓
3	2	5	✓
4	12	24	
5	72	132	
6	480	840	
7	3600	6120	
8	30240	50400	
9			
10			
11			
12	399,168,000	636,668,800	

Recently an improved estimate has been derived by Chris McCord using the reflection symmetry as well as the rotational one. He finds at least

$$n!\left(\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}\right) \text{ planar, non collinear CC's}$$

assuming non-degeneracy. This is roughly log(n) times Palmore's estimate.

Thus for even moderately large n, there is a huge number of periodic orbits arising from the planar CC's (assuming non-degeneracy). Even though these solutions are very special, they may have important effects on the dynamics of the n-body problem by several mechanisms. First there may be many other periodic and quasi-periodic orbits nearby due to KAM theory. Second, if the orbits are unstable they may have stable and unstable manifolds which disperse throughout the phase space with the result that many other orbits pass near the CC orbits. Third, when the angular momentum is low orbits may pass near collision at the center of mass and, as we will see later, the CC's play a dominant role there.

Exact Counts of CC's We will now describe a method due to Xia by which one can arrive at exact counts of the numbers of CC's for certain special choices of the masses. Suppose the masses are $m_1, \cdots, m_{n-1}, m_n = \varepsilon$ where ε is a small parameter and suppose that q^ε is a sequence of CC's which converges as $\varepsilon \to 0$ to a limiting configuration $\bar{q}$. One can show that q_i^ε keep away from one another during the limiting process. Taking the limit of the CC equations gives

$$\sum_{\substack{j\neq i\\ j=1}}^{\overset{\text{note}}{n-1}} \frac{m_j(\bar{q}_j-\bar{q}_i)}{\bar{r}_{ij}^{\,3}} + \lambda \bar{q}_i = 0 \qquad i = 1, \cdots, n-1$$

$$\sum_{j=1}^{n-1} \frac{m_j(\bar{q}_j-\bar{q}_n)}{\bar{r}_{jn}^{\,3}} + \lambda \bar{q}_n = 0.$$

Thus $(\bar{q}_1, \cdots, \bar{q}_{n-1})$ is a CC of the $(n-1)$ body problem. Knowing these, the last equation becomes

$$\nabla V(\bar{q}_n) = 0$$

where

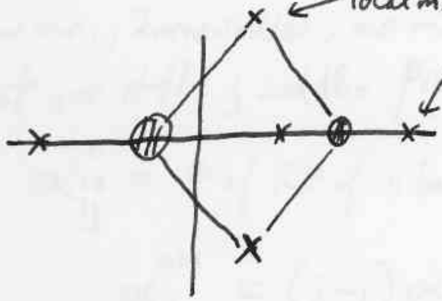
$$V(x) = \sum_{j=1}^{n-1} \frac{m_j}{|\bar{q}_j - x|} + \frac{1}{2}\lambda |x|^2 \qquad x \in \mathbb{R}^2$$

is called the *effective potential*. Thus the problem splits into the $(n-1)$-body RE problem and the problem of finding critical points of a function on $\mathbb{R}^2$ (this is not really a trivial problem!)

For example when $n=3$ there are 2 nonzero masses which obviously form a line segment. If we take $\lambda=1$ and normalize the masses to $m_1=1-\mu$, $m_2=\mu$ we find $\bar{q}_1 = \begin{bmatrix}-\mu\\0\end{bmatrix}$ $\bar{q}_2 = \begin{bmatrix}1-\mu\\0\end{bmatrix}$ and

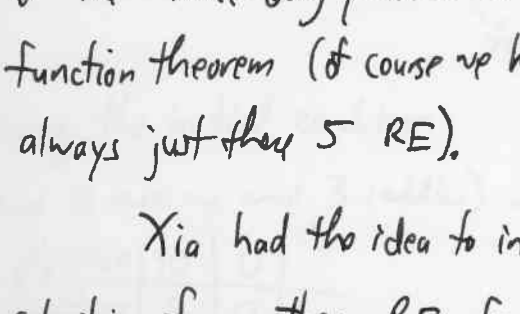
$$V(x) = V(x_1,x_2) = \frac{1-\mu}{\sqrt{(x_1+\mu)^2+x_2^2}} + \frac{\mu}{\sqrt{(x_1+\mu-1)^2+x_2^2}} + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 .$$

This has 5 critical points representing the 5 possible positions of the 3rd mass



These correspond to the 5 RE of the general 3 body problem (Euler's and Lagrange's). Now if we put a 3rd mass at one of these points it can be continued to a real RE of the three-body problem for all m_3 sufficiently small using the implicit function theorem (of course we know that m_3 need not be small, there are always just these 5 RE).

Xia had the idea to inductively construct RE for $n=4$ and higher starting from these RE for $n=3$, m_3 small. Suppose we choose to put the third mass at an equilateral Lagrange point. The resulting three-body RE will have a new effective potential with a new singularity at the Lagrange point. Xia showed that for m_3 small there are 4 new critical points produced nearby the new singularity; meanwhile, the old critical points continue.



There are 2 new local minima and 2 new saddle points. If we had put the third mass at one of the collinear Euler points, the new effective potential has just 2 new critical points, both saddles.

Now we can put a 4th mass at one of these new points and continue the construction. It can be shown that if m_3 is suff small and m_4 is suff small relative to m_3, etc., then <u>all</u> RE for these masses arise from the construction. Thus we can just count how many RE are produced.

Say that a RE is of type (i,j) if the effective potential has exactly i local minima and j saddles (one can show that V is a superharmonic function, so it has no maxima. Compare Palmore's index estimate). If we add a new mass at one of the i local minima we get an RE of type $(i+1, j+2)$ (2 new min., 2 new saddles, one min used up by the new mass). Similarly a new mass at one of the j saddles produces a RE of type $(i, j+1)$. If we let α^n_{ij} = # of RE of n bodies of type (i,j) then we have

$$\alpha^{n+1}_{i,j} = \underset{\substack{\uparrow \\ \text{from local minima}}}{(i-1)\,\alpha^{n}_{i-1,j-2}} + \underset{\substack{\nwarrow \\ \text{from saddles}}}{(j-1)\,\alpha^{n}_{i,j-1}}$$

Using the initial condition $\alpha^2_{2,3} = 1$ (one RE of 2-body problem and it has 2 minima and 3 saddles) we can generate a table of α^n_{ij}. Actually, $n = j-i+1$ so the n-body RE's lie along a 45° line in the table. For example there are $2+3=5$ RE for $n=3$ and $12+16+6=34$ for $n=4$. The numbers obtained for larger n vastly exceed the Morse theory estimates (but we are looking at special masses).

8	2520	960	90	0
7	360	120	6	0
6	60	16	0	0
5	12	2	0	0
4	3	0	0	0
3	1	0	0	0
	2	3	4	i

5body

4body

3body

n	#RE ($\Sigma\alpha_{ij}$)	Morse Th.
3	5	5
4	34	24
5	294	132
6	3,096	840
7	38,520	6,120
8	553,680	50,400
9	9,036,720	463,680
10	165,191,040	4,717,440

Tien showed that the total produced by this method is (for n bodies)

$$T_n = (n-2)!\left((n-2)2^{n-1}+1\right)$$

(Morse theory was $(n-1)!\left(\frac{3n-4}{2}\right)$). A similar study for CC's (in $\mathbb{R}^3$) gives

$$T_n = \frac{(n-2)!}{16}\left((2n+3)3^{n-1} + 2n^2 - 6n - 1\right).$$

Thus there are many more CC's than there are RE's in this case.

Stability

We will try to analyze the stability of Lagrange's circular periodic solutions associated to a relative equilibrium of the n-body problem. Recall that a relative equilibrium is a planar CC, that is, a planar solution of

$$M^{-1}\nabla U(q) + \lambda q = 0 \qquad \lambda > 0$$

$q \in P = \{ q \in \mathbb{R}^{3n} \setminus \Delta : q_i \in \mathbb{R}^2 ; i = 1, \cdots, n \}$. Given a relative equilibrium (abbreviated RE) there is a solution of the n-body problem of the form $A(t)q$ where $A(t) = \begin{bmatrix} \cos(kt) & -\sin(kt) & 0 \\ \sin(kt) & \cos(kt) & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $k^2 = \lambda = \dfrac{U(q)}{q^T M q}$.

The initial momentum of this solution is

$$\overset{T}{p} = M\mathbb{K}q = \mathbb{K}Mq$$

where $\mathbb{K}$ is the $3n \times 3n$ matrix which is block-diagonal and has 3×3 blocks $\begin{bmatrix} 0 & -k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The angular momentum of the Lagrangian solution is

$$\Omega = \omega_0 = \begin{bmatrix} 0 \\ 0 \\ k \cdot q^T M q \end{bmatrix}$$

and its energy is

$$H = \tfrac{1}{2}k^2 \cdot q^T M q - U(q) = q^T M q \cdot \left(\tfrac{1}{2}k^2 - k^2\right) = -\tfrac{1}{2}k^2 \cdot q^T M q = h.$$

The scale invariant parameter $h|\omega_0|^2$ is: $h|\omega_0|^2 = -\frac{1}{2}U(q)^2 q^T M q$.

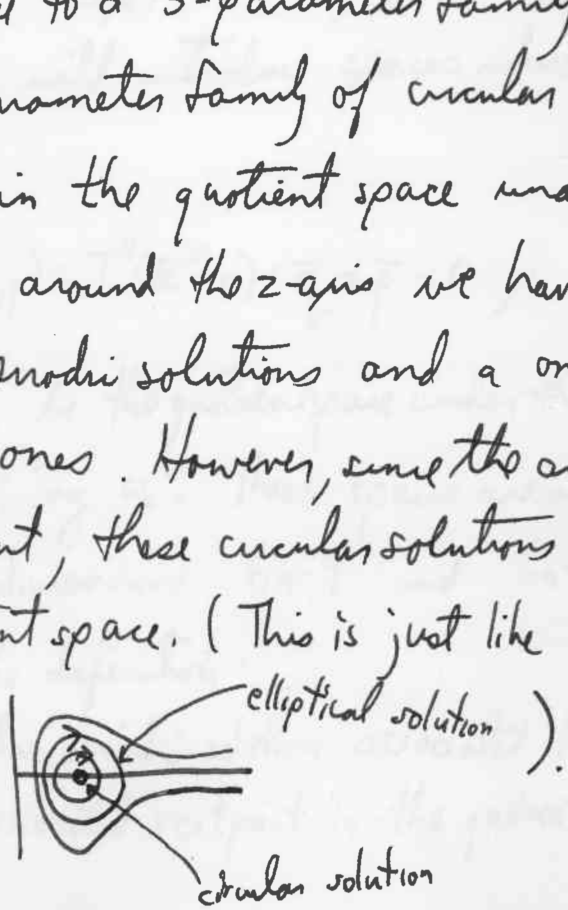
Of course q also gives rise to other homographic periodic solutions of the form: $r(t)\,A(t)\,q$ where $A = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and where $r(t), \theta(t)$ solve the Kepler problem.

These can be distinguished from one another by their energies and angular momenta. If we fix any negative energy $h<0$ then for any angular momentum $\begin{bmatrix} 0 \\ 0 \\ \omega_{03} \end{bmatrix}$ with

$$|h|\,|\omega_0|^2 \leq \tfrac{1}{2}U(q)^2 q^T M q$$

there will be a solution of this type with that energy and angular momentum. Equality above corresponds to the circular solution. One can also choose the initial angle θ_0. Thus each RE gives rise to a 3-parameter family of periodic solutions and a 2-parameter family of circular periodic solutions. If we work in the quotient space under the action of the rotations around the z-axis we have a 2-parameter family of periodic solutions and a one parameter family of circular ones. However, since the angle θ has been quotiented out, these circular solutions become restpoints in the quotient space. (This is just like the reduced Kepler problem:



).

This change in topology from a circle to a point arises from the fact that RE's are also critical configurations. Recall that for each such q there is a momentum p such that the integrals of the n-body problem fail to be independent at (q,p). This associated momentum, p, is exactly the one which gives rise to the circular periodic orbits. Thus the critical level sets $S_{(\omega_0,h)}$ are exactly those containing one of the Lagrangian periodic solutions.

We want to work in a quotient space under the symmetry so we can study a restpoint instead of a periodic orbit. Yet the quotient spaces $S'_{(\omega_0,h)}$ containing the restpoints are not smooth manifolds. We will introduce spaces where Ω is fixed but H is not.

Let

$$S_{\omega_0} = \{(q,p) \in T^*(\mathbb{R}^{3n} \setminus \Delta) : \bar{q} = \bar{p} = 0 \ , \quad (q,p) = \omega_0 \in \mathbb{R}^3\}$$

and let S'_{ω_0} be the quotient space under the 1-parameter group of rotations fixing ω_0. These spaces are always smooth manifolds of dimensions $6n-9$ and $6n-10$ respectively. Then make the definition:

> The circular periodic solution associated to a RE is stable if the associated restpoint in the quotient space S'_{ω_0} is stable.

Here we mean the usual definition for restpoint stability in mechanics:

Given any nbhd $\mathcal{U}$ of (q,p) there is a nbhd V such that if $(q',p') \in V$ then $(q(t),p(t)) \in \mathcal{U}$ for all $t \in \mathbb{R}$.

It is convenient to introduce rotating coordinates. Let k be the rotational velocity of the circular solution and let $A(t)$ be the rotation matrix

$$A(t) = \begin{bmatrix} \cos(kt) & -\sin(kt) & 0 \\ \sin(kt) & \cos(kt) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Introduce new coordinates: $Q_i = A(t) q_i$, $P_i^T = A(t) p_i^T$.

then the differential equations become:

$$\dot{Q} = M^{-1} P^T + \mathbb{K} Q$$

$$\dot{P}^T = \nabla U(Q) + \mathbb{K} P^T$$

where $\mathbb{K}$ is (as above) the $3n \times 3n$ matrix with blocks $\begin{bmatrix} 0 & -k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

This is Hamiltonian with

$$H(Q,P) = \tfrac{1}{2} P M^{-1} P^T + P \mathbb{K} Q - U(Q).$$

The Lagrangian circular periodic solution is now a restpoint (or rather a whole circle of restpoints). Indeed, setting $\dot{Q} = \dot{P} = 0$ gives

$$P^T = -MKQ$$

$$\nabla U(Q) = -M\mathbb{K}^2 Q$$

Now K^2 has blocks $\begin{bmatrix} -k^2 & 0 & 0 \\ 0 & -k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so we recover the RE equation with constant $\lambda = k^2$.

In these coordinates, the center of mass and total momentum equations are

$$\overline{Q} = \frac{1}{m} \sum_{i=1}^{n} m_i Q_i = 0 \quad \text{and} \quad \overline{P} = \sum_{i=1}^{n} P_i = 0.$$

The angular momentum equation is

$$\Omega(Q,P) = \sum_{i=1}^{n} Q_i \times P_i^T = \omega_0 = \begin{bmatrix} 0 \\ 0 \\ \omega_{03} \end{bmatrix}.$$

It is extremely difficult to determine stability of restpoints of Hamiltonian systems. One of the few feasible methods is to show that the Hamiltonian (or some other integral) has a maximum or minimum at the restpoint. If the max/min is isolated then H is a Lyapunov function so the restpoint is stable. We will now show that this approach never works for RE's. We will show that $H|_{S_{\omega_0}}$ does not have a maximum or a minimum at (Q,P). We will find tangent vectors $u_1, u_2 \in T_{(Q,P)} S_{\omega_0}$ such that moving in the u_1 direction increases H while moving in the u_2 direction decreases H. Take u_1 of the form $\delta Q = 0$ $\delta P \neq 0$. A vector of this form is tangent to S_{ω_0} provided

$$\sum Q_i \times \delta P_i^T = 0.$$

Clearly there is some vector of this form and since the kinetic energy $\frac{1}{2} P M^{-1} P^T$ is positive definite, H increases in this direction. Similarly we will take u_2 of the form $\delta P = 0$, $\delta Q \neq 0$, with

$$0 = \sum \delta Q_i \times P_i^T = \sum m_i \, \delta Q_i \times \begin{bmatrix} 0 & -k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q_i^\circ = k \sum m_i \, \delta Q_i \cdot Q_i$$

This is just the tangent space to the normalized configuration space S (assuming without loss of generality that $\langle Q, Q \rangle = 1$). Now we know that as a critical point of $U|_S$, Q has Morse index at most $(n-2)$. It follows that $U(Q)$ cannot have a local maximum at Q. Choose δQ which makes $U(Q)$ increase. Then $H(Q,P)$ decreases.

For this reason, it is hopeless to try to prove stability of RE's. Instead we will consider linear stability. The linearization of the n-body problem in rotating coordinates is $\dot{v} = Lv$ where:

$$L = \begin{array}{c|c|c|} & 3n & 3n \\ \hline 3n & K & M^{-1} \\ \hline 3n & D^2U(Q) & K \end{array}$$

This is a linear Hamiltonian system with Hamiltonian $\frac{1}{2}v^T A v = Q(v)$

$$A = \left[\begin{array}{c|c} -D^2U(Q) & K \\ \hline K^T & M^{-1} \end{array}\right]$$

(We will treat V as an element of $\mathbb{R}^{6n}$ instead of $\mathbb{R}^{3n} \times \mathbb{R}^{3n*}$).

Note that $A^T = A$ and $L = -JA$ where

$$J = \begin{array}{c|c|c|} & 3n & 3n \\ \hline 3n & 0 & -I \\ \hline 3n & I & 0 \end{array}.$$

A matrix of this form ($-J$ times symmetric) is called Hamiltonian.

Let L be a Hamiltonian matrix and let $P_L(\lambda)$ be its characteristic polynomial. Then using $J^2 = -I$ and $J = -J^T$ and $|J| = 1$ we have:

$$P_L(\lambda) = |L - \lambda I| = |-JA + \lambda J^2| = \overset{1}{\cancel{|-J|}}\,|A - \lambda J| = |A^T + \lambda J| = |A + \lambda J| = P_L(-\lambda).$$

It follows that if λ is an eigenvalue then so is $-\lambda$. Moreover, since L is real, also $\bar{\lambda}$ and $-\bar{\lambda}$ are eigenvalues. There are several possibilities

$\lambda = 0$ $\quad \lambda = \bar{\lambda} = -\lambda = -\bar{\lambda}$ trivial eigenvalue

$\lambda = a \in \mathbb{R} \setminus 0$ $\quad \lambda = \bar{\lambda} = a$, $-\lambda = -\bar{\lambda} = -a$ real pair

$\lambda = ib, b \in \mathbb{R}\setminus 0$ $\quad \lambda = -\bar{\lambda} = ib$, $-\lambda = \bar{\lambda} = -ib$ imag. pair

$\lambda = a+ib,\ ab \neq 0$ $\quad \bar{\lambda} = a-ib$, $-\lambda = -a-ib$, $-\bar{\lambda} = -a+ib$ complex quadruple.

Note that if there is a real pair or a complex quadruple then $\dot{v} = Lv$ has exponentially growing solutions and the system is unstable. The only hope for stability is to have imaginary (possibly 0) eigenvalues. We will call a restpoint of a Hamiltonian system <u>spectrally stable</u> if its linearization, L, has all imaginary eigenvalues. We will call the restpoint <u>linearly stable</u> if $v = 0$ is a stable restpoint for the linearized equation $\dot{v} = Lv$. This implies spectral stability but, in addition, requires that there be no nontrivial Jordan blocks in the Jordan canonical form of L.

To develop the theory of linear Hamiltonian systems further one needs the symplectic form or skew inner product

$$\omega(v,w) = v^T J w$$

(a different ω than the angular momentum). Note

$$\omega(w,v) = w^T J v = v^T J^T w = -\omega(v,w).$$

Suppose L is a $2m\times 2m$ Hamiltonian matrix and let $W \subset \mathbb{C}^{2m}$ be an invariant subspace : $L(W) \subset W$. Then $W^\perp = \{v \in \mathbb{C}^{2m} : \omega(v,w)=0 \text{ for all } w\in W\}$ is also invariant. To see this, suppose $v \in W^\perp$. Then for $w \in W$

$$\omega(Lv,w) = v^T L^T J w = -v^T A^T J^T J w = -v^T A w$$

$$= v^T J^2 A w = -\omega(\underset{\overset{\nearrow}{\text{in } W^\perp}}{v}, \underset{\overset{\uparrow}{\text{in } W}}{Lw}) = 0$$

so $Lv \in W^\perp$ as required.

Next consider an eigenvector or generalized eigenvector of L, that is, $v \in \mathbb{C}^{2m}$ with $(L-\lambda I)^k v = 0$ for some $k \geq 1$. We will show

If v and w are generalized eigenvectors of L with eigenvalues λ and μ, then if $\lambda + \mu \neq 0$ we have $\omega(v,w)=0$.

In other words gen. eigenvectors are skeworthogonal unless $\lambda = -\mu$.

The proof makes use of a formula which is valid for any $v \in \mathbb{C}^{2m}$, $\lambda \in \mathbb{C}$:

$$\omega(v,(L+\lambda I)w) = -\omega((L-\lambda I)v, w)$$

(To prove it just use $L = -JA$, $A = A^T$ and $J = -J^T$, $J^2 = -I$ to simplify both sides.) Now if $\mu \neq -\lambda$ then $(L+\lambda I)$ restricts to an isomorphism of the generalized eigenspace for μ. Hence any w which is a gen. eigenvector for μ can be written $w = (L+\lambda I)^k w'$ for some w'. Hence if $(L-\lambda I)^k v = 0$:

$$\omega(v,w) = \omega(v,(L+\lambda I)^k w') = (-1)^k \omega((L-\lambda I)^k v, w') = 0.$$

Next we will discuss a simple sufficient condition for linear stability.

Suppose L is a $2m \times 2m$ real Hamiltonian matrix. Suppose that every eigenvector $v \in \mathbb{C}^{2m}$ of L satisfies $\omega(v,\bar{v}) \neq 0$. Then 0 is stable for the differential equation $\dot{v} = Lv$.

Note that the hypothesis refers only to eigenvectors, not generalized eigenvectors. Indeed, if 0 is stable there are no nontrivial generalized eigenvectors, but one need not know this in advance.

We need to show that all eigenvalues are imaginary and that there are no nontrivial gen. eigenvectors. Suppose $Lv = \lambda v$, with $\lambda = a + ib$. Then $L\bar{v} = \bar{\lambda}\bar{v}$. If λ were real we could take v real as well and then $\omega(v,\bar{v}) = \omega(v,v) = 0$, a contradiction. Thus $b \neq 0$. On the other hand, $\omega(v,\bar{v}) \neq 0$ implies $\lambda + \bar{\lambda} = 0$ so $a = 0$. Hence $\lambda = ib \neq 0$. Next suppose there were a gen. eigenvector w for λ which is not an eigenvector. We may assume $(L-\lambda I)^2 w = 0$ and $(L - \lambda I)w = v \neq 0$. Then v is an eigenvector for λ and

$$\omega(v,\bar{v}) = \omega((L-\lambda I)w, \bar{v}) = -\omega(w, (L+\lambda I)\bar{v}) \underset{\underset{\bar{\lambda}=-\lambda}{\nearrow}}{=} -\omega(w,(L-\bar{\lambda}I)\bar{v}) = 0$$

which is a contradiction.

There is an interesting and equivalent formulation of the condition $\omega(v,\bar{v}) \neq 0$ which is expressed in terms of the real invariant subspace associated to the eigenvalue pair $\lambda, \bar{\lambda}$. Let $E_\lambda \subset \mathbb{C}^{2m}$ be the eigenspace of λ and let $R_\lambda = \{ \mathrm{Re}(v) : v \in E_\lambda \}$ be the space of real parts of eigenvectors. If $\dim_{\mathbb{C}} E_\lambda = k$ then $\dim_{\mathbb{R}} R_\lambda = 2k$. Moreover if we repeat the construction with $\bar{\lambda}$ we have $R_\lambda = R_{\bar{\lambda}}$. Now if $v \in E_\lambda$, $v = Re(v) + i\,Im(v)$ we have

$$\omega(v, \bar{v}) = \omega(Re(v) + i\,Im(v), Re(v) - i\,Im(v)) = 2i\,\omega(Im(v), Re(v)).$$

Now consider the value of the quadratic Hamiltonian $Q(v) = \frac{1}{2} v^T A v$ on $Re(v)$. We have $L = -JA$ so $A = JL$ and $L(Re(v)) = a\,Re(v) - b\,Im(v)$ if $\lambda = a + ib$. Hence

$$Q(Re(v)) = \frac{1}{2} Re(v)^T J(a\,Re(v) - b\,Im(v)) = \frac{b}{2}\,\omega(Im(v), Re(v))$$

Thus $\omega(v, \bar{v}) \neq 0$ on E_λ if and only if $Q \neq 0$ on R_λ. In other words the Hamiltonian is definite (either + or −) on the subspace R_λ. The sign of Q on R_λ will be called the Hamiltonian sign of $\lambda, \bar{\lambda}$. We know from the discussion above that if $\lambda, \bar{\lambda}$ have a well-defined Hamiltonian sign then $\lambda = ib$, $b \neq 0$ and $\bar{\lambda} = -\lambda$.

A linear Hamiltonian system such that $\omega(v, \bar{v}) \neq 0$ for every eigenvector v (or equivalently, such that every eigenvalue has a well-defined Hamiltonian sign) is called *strongly stable* or *parametrically stable*. Since the condition $\omega(v, \bar{v}) = 0$ for some eigenvector is a closed condition in matrix space, Strong stability holds for an open set of Hamiltonian matrices. Thus it persists under small perturbations or changes of parameters

After this excursion into the general theory of linear Hamiltonian systems, we will return to the study of the matrix:

$$L = \begin{array}{c} \left[\begin{array}{c|c} \mathbb{K} & M^{-1} \\ \hline D^2U(Q) & \mathbb{K} \end{array}\right]\begin{array}{l} 3n \\ 3n \end{array} \\ \begin{array}{cc} 3n & 3n \end{array} \end{array}$$

Q a relative equilibrium in $\mathbb{R}^2$

$$\mathbb{K} = \left[\begin{array}{c|c|c} * & & \\ \hline & * & \\ \hline & & * \end{array}\right] \longleftarrow \begin{bmatrix} 0 & -k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

k = angular velocity
of circular solution

Recall that when defining stability for RE's we restricted to the quotient space S'_{ω_0} of the space with $\overline{Q} = \overline{P} = 0$, $\Omega(Q,P) = \omega_0 = \begin{bmatrix} 0 \\ 0 \\ \omega_0 \end{bmatrix}$. It is necessary to do something similar when working on linear stability.

To eliminate the center of mass and total momentum, let E_1 denote the subspace of $\mathbb{C}^{6n}$ spanned by the 6 vectors:

$$v_k = \left[\begin{array}{c} e_k \\ \vdots \\ e_k \\ \hline 0 \end{array}\right]\begin{array}{l} 3n \\ \\ 3n \end{array} \quad w_k = \left[\begin{array}{c} 0 \\ \hline m_1 e_k \\ \vdots \\ m_n e_k \end{array}\right]\begin{array}{l} 3n \\ \\ 3n \end{array} \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad k = 1,2,3.$$

Clearly these are vectors associated to translations of the position and momentum variables. It is easy to check that E_1 is invariant under L (and also under J). The matrix of $L|_{E_1}$ in the basis $\{v_1, v_2, w_1, w_2, v_3, w_3\}$ is

$$L|_{E_1} = \left[\begin{array}{cc|cc|cc} 0 & -k & 1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ k & 0 & 0 & 1 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & -k & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ & & k & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & & & 0 & 0 \end{array}\right].$$

This would lead to instability due to the nontrivial Jordan blockstructure if we did not exclude E_1 in our definition. Intuitively, these Jordan blocks are associated to the drift which would occur if we allowed nonzero total momentum.

A similar problem occurs with the angular momentum. Let E_2 be the subspace of $\mathbb{C}^{6n}$ spanned by the 8 vectors:

$$V_0 = \begin{bmatrix} Q \\ \hline 0 \end{bmatrix}\begin{matrix} 3n \\ 3n \end{matrix} \quad w_0 = \begin{bmatrix} 0 \\ \hline MQ \end{bmatrix}\begin{matrix} 3n \\ 3n \end{matrix} \qquad V_k = \begin{bmatrix} K_k Q \\ \hline 0 \end{bmatrix}\begin{matrix} 3n \\ 3n \end{matrix} \quad W_k = \begin{bmatrix} 0 \\ \hline K_k MQ \end{bmatrix}\begin{matrix} 3n \\ 3n \end{matrix} \qquad k = 1,2,3$$

where K_k are $3\times 3n$ matrices with 3×3 diagonal blocks:

$$K_1 : \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad K_2 : \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad K_3 : \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(If Q is collinear in $\mathbb{R}^1$, then $V_1 = W_1 = 0$, and $\dim(E_2) = 6$ not 8).

Clearly E_2 is associated to dilation and rotation of the coordinates and momenta. Once again, E_2 is both L and J invariant. The matrix of $L|_{E_2}$ in the basis $\{V_0, V_3, W_0, W_3, V_1, W_1, V_2, W_2\}$ is

$$L|_{E_2} = \left[\begin{array}{cccc|cc|cc} 0 & -k & 1 & 0 & & & & \\ k & 0 & 0 & 1 & & 0 & & \\ 2k^2 & 0 & 0 & -k & & & & \\ 0 & -k^2 & k & 0 & & & & \\ \hline & & & & 0 & 1 & & \\ & & & & -k^2 & 0 & \multicolumn{2}{c}{0} \\ \cline{5-8} & 0 & & & & & 0 & 1 \\ & & & & \multicolumn{2}{c|}{0} & -k^2 & 0 \end{array}\right].$$

(To derive this one uses the facts that:

$$D^2U \cdot Q = -2\nabla U(Q) = 2\frac{U}{q^T M q} MQ = 2k^2 MQ$$

$$D^2U \cdot K_k Q = -k^2 M K_k Q$$

and the fact that Q is planar.)

The lower right 2×2 blocks give rise to the repeated eigenvalue $\lambda = \pm ik$.
(in the collinear case, it is not repeated)
For the upper block one can use the following result (which will be used again later):

The characteristic polynomial of the 4×4 matrix

$$\begin{bmatrix} 0 & -k & 1 & 0 \\ k & 0 & 0 & 1 \\ ak^2 & 0 & 0 & -k \\ 0 & bk^2 & k & 0 \end{bmatrix}$$

is $\lambda^4 + (2-a-b)k^2\lambda^2 + (1+a)(1+b)k^4$.

Applying this to the upper left block of $L|_{E_2}$ $(a=2, b=-1)$ gives

$$\lambda^2(\lambda^2+k^2)=0$$

so we have another copy of $\lambda = \pm ik$ and a double $\lambda = 0$. The latter has a nontrivial Jordan block structure which would again be fatal for stability if not excluded by fiat. Intuitively the double $\lambda=0$ arise from the third component of angular momentum and from the rotation symmetry around the z-axis within S_{ω_0}. These are just the factors which are excluded by fixing ω_{03} and working in the quotient space S'_{ω_0}.

Thus we should formulate a definition of linear stability which excludes consideration of E_1 and of the $\lambda=0$ eigenspace in E_2. We are not justified in ignoring the $\lambda = \pm ik$ eigenspaces in E_2 however

But note that these eigenvalues have a well defined-Hamiltonian sign (+). To see this note that in this basis J is represented by

$$J = \left[\begin{array}{cc|c|c} 0 & -I & & \\ \hline I & 0 & & \\ \hline & & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & \\ \hline & & & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \end{array}\right] \cdot Q^TMQ$$

which makes that matrix of the quadratic Hamiltonian $Q(v)$

$$A = JL = \left[\begin{array}{c|c|c} \begin{matrix} -2k^2 & 0 & 0 & k \\ 0 & k^2 & -k & 0 \\ 0 & -k & 1 & 0 \\ k & 0 & 0 & 1 \end{matrix} & & \\ \hline & \begin{matrix} k^2 & 0 \\ 0 & 1 \end{matrix} & \\ \hline & & \begin{matrix} k^2 & 0 \\ 0 & 1 \end{matrix} \end{array}\right] Q^TMQ$$

The lower two blocks are clearly positive definite. The eigenvector of $\lambda = ik$ for the upper block has $v = \begin{bmatrix} 1 \\ -2i \\ -ki \\ k \end{bmatrix}$ and

$$Q(Re(v)) = \tfrac{1}{2} k^2 \cdot Q^TMQ > 0$$ as claimed.

Thus is discussion motivates the following definition. Introduce the subspace $E = (E_1 \oplus E_2)^\perp$ where $\perp$ denotes the skew-orthogonal complement, that is, $E = \{ v \in \mathbb{C}^{6m} : \omega(v,w) = 0$ for all $w \in E_1 \oplus E_2 \}$. Then since J restricts to a non-degenerate form on $E_1 \oplus E_2$ (that is $E_1 \oplus E_2 \cap (E_1 \oplus E_2)^\perp = \{0\}$) there is a direct sum decomposition

$$\mathbb{C}^{6m} = E_1 \oplus E_2 \oplus E$$

. Moreover E is invariant for L since E_1 and E_2 are.

We will call a RE Q <u>linearly stable</u> if 0 is stable for $L|_E$ and <u>strongly linearly stable</u> if, in addition, the Hamiltonian sign of $\lambda = \pm ik$ for $L|_E$ (if it occurs as an eigenvalue of $L|_E$ at all) is positive.

There is another useful splitting of L arising from the fact that Q is a planar configuration. Recall that for a planar CC, $D^2U(Q)$ takes the form

$$D^2U(Q) = \begin{array}{cc} \left[\begin{array}{c|c} D' & \\ \hline & B \end{array}\right] & \begin{array}{c} 2n \\ n \end{array} \\ \begin{array}{cc} 2n & n \end{array} & \end{array}$$

where we have regrouped the 3n coordinates as $Q = (x_1, \cdots, x_n, y_1 \cdots y_n, z_1 \cdots z_n)$. Here $B_{ij} = \frac{m_i m_j}{r_{ij}^3}$, $B_{ii} = -\sum_{j \neq i} B_{ij}$ is the matrix which was studied in some detail above. The eigenvalues of $M^{-1}B$ are $\mu = 0, \mu_2, \mu_3 \cdots \mu_n$ where $\mu_k < 0$ for $k = 2, \cdots, n$.*

Suppose that u_k is an eigenvector of $M^{-1}B$ for eigenvalue μ_k.

Let

$$V_k = \begin{bmatrix} 0 \\ 0 \\ u_1 \\ \vdots \\ 0 \\ 0 \\ u_n \\ \hline 0 \end{bmatrix} \begin{array}{c} 3n \\ \\ 3n \end{array} \qquad W_k = \begin{bmatrix} 0 \\ \hline 0 \\ 0 \\ m_1 u_1 \\ \vdots \\ 0 \\ 0 \\ m_n u_n \end{bmatrix} \begin{array}{c} 3n \\ \\ 3n \end{array}$$

* In the collinear case $\mu_2 = -\frac{U(Q)}{Q^T M Q} = -k^2$. In the non-collinear case $\mu_2 = \mu_3 = -k^2$ (from symmetry). These are from E_2 above.

Then v_k and w_k span a 2-dimensional invariant subspace for L on which L has matrix

$$\begin{bmatrix} 0 & 1 \\ \mu_k & 0 \end{bmatrix}$$

and the quadratic Hamiltonian has matrix

$$\begin{bmatrix} -\mu_k & 0 \\ 0 & 1 \end{bmatrix}$$

Together, these generate the $2n$-dimensional subspace of $\mathbb{C}^{6m}$ of vectors with only z-components and so on this space we find that L has eigenvalues $0, 0, \pm i\sqrt{|\mu_k|}$; $k = 2, \cdots, n$. Since these subspaces are mutually skew-orthogonal, the Hamiltonian is the sum of the individual Hamiltonians and so the Hamiltonian signs of all the non-zero eigenvalues are $+$.

Thus we are reduced to studying $L|_{E'}$ where $E' = E \cap \mathbb{C}^{4m}$ and $\mathbb{C}^{4n}$ means the planar vectors. (For strong stability one should also check that Hamiltonian signs of any repeated eigenvalues agree). The matrix of L on $\mathbb{C}^{4m}$ is

$$L|_{\mathbb{C}^{4m}} = \left[\begin{array}{c|c} \mathbb{K} & M^{-1} \\ \hline D' & \mathbb{K} \end{array}\right]$$

where now $\underline{\mathbb{K}} = \left[\begin{array}{c|c|c} \overset{2}{\begin{matrix} 0 & -k \\ k & 0 \end{matrix}} & 0 & \\ \hline 0 & \begin{matrix} 0 & -k \\ k & 0 \end{matrix} & \\ \hline & & \ddots \end{array}\right]$

$$M = \begin{bmatrix} m_1 & & & & \\ & m_1 & & & \\ & & m_2 & & \\ & & & m_2 & \\ & & & & \ddots \end{bmatrix}$$

are $2n \times 2n$.

Instability of the Collinear RE's If Q is collinear we have the further splitting $D' = \left[\begin{array}{c|c} -2B & 0 \\ \hline 0 & B \end{array}\right]$. Suppose, as above, that

$M^{-1}Bu_k = \mu_k u_k$. Then the vectors

$$V_1 = \begin{bmatrix} u_1 \\ 0 \\ u_2 \\ 0 \\ \vdots \\ \hline 0 \end{bmatrix} \begin{array}{l} \\ \\ 2n \\ \\ \\ \\ 2n \end{array} \quad V_2 = \begin{bmatrix} 0 \\ u_1 \\ 0 \\ u_2 \\ \vdots \\ \hline 0 \end{bmatrix} \begin{array}{l} \\ \\ 2n \\ \\ \\ \\ 2n \end{array} \quad w_1 = \begin{bmatrix} 0 \\ \hline m_1u_1 \\ 0 \\ m_2u_2 \\ 0 \\ \vdots \end{bmatrix} \begin{array}{l} 2n \\ \\ \\ 2n \\ \\ \\ \end{array} \quad W_2 = \begin{bmatrix} 0 \\ \hline 0 \\ m_1u_1 \\ 0 \\ m_2u_2 \\ \vdots \end{bmatrix} \begin{array}{l} 2n \\ \\ \\ 2n \\ \\ \\ \end{array}$$

span a four dimensional invariant subspace on which L has matrix (using basis $\{v_1, v_2, w_1, w_2\}$)

$$\left[\begin{array}{cc|cc} 0 & -k & 1 & 0 \\ k & 0 & 0 & 1 \\ \hline -2\mu_k & 0 & 0 & -k \\ 0 & \mu_k & k & 0 \end{array}\right]$$

Note: In the collinear case we know that the eigenvalues μ_k satisfy:

$$\mu_2 = \frac{-U(q)}{q^TMq} = -k^2$$

$$\mu_k < \mu_2 \text{ for } k \geq 3.$$

Applying our lemma about such matrices gives the characteristic polynomial

$$\lambda^4 + (2-\alpha_k)k^2\lambda^2 + (1+2\alpha_k)(1-\alpha_k)k^4$$

where $\alpha_k = \frac{-\mu_k}{k^2}$. We have $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_k > 1$; $k \geq 3$. The first two give eigenvalues $\lambda = 0, 0$ and $\pm ik$. These are from the subspaces E_1 and E_2 which we are excluding. Since $\alpha_k > 1$, for $k \geq 3$ we find a positive and a negative value for λ^2. The former leads to real eigenvalues and hence to instability.

Even in a case of instability, knowledge of the structure of the linearized flow can reveal interesting information about the orbits of the n-body problem near the circular periodic solution.

Consider the collinear RE solutions as restpoints of the flow in the (6n-10)-dimensional manifold S_{ω_0}'. The eigenvalues of its linearization will be the 6n eigenvalues found above with the following exceptions:

$0, 0, \pm ik$ from the non-planar block

$0, 0, \pm ik, \pm ik$ from the planar block (leaving one $\pm ik$ still)

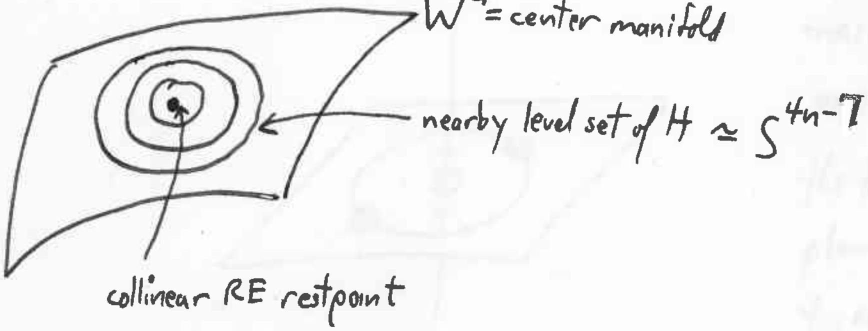
This leaves $(2n-4)$ eigenvalues

$\lambda = \pm i \sqrt{|\mu_k|}$ $k = 3, \cdots, n$, in the non-planar directions

$(2n-4)$ imaginary eigenvalues in the planar directions

$(n-2)$ positive real eigenvalues and $(n-2)$ negative real eigenvalues in the planar directions and the pair $\pm ik$ in the planar directions.

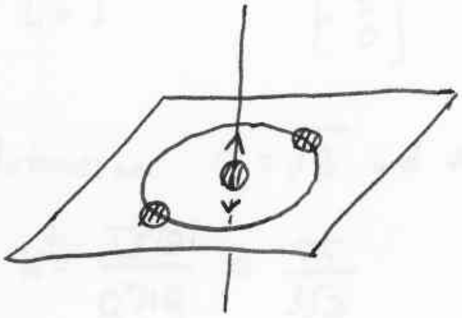
Now the center manifold theorem shows that there is a $(4n-6)$ dimensional, locally invariant manifold containing the restpoint and tangent to the imaginary eigenspaces. Normally it is difficult to conclude much about the dynamics on a center manifold. However, in this case, we can use conservation of energy to say more. We saw that the Hamiltonian sign of the non-planar imaginary eigenvalues was +. It turns out to be the same for the planar imaginary eigenvalues (the proof is just to find the eigenvectors and check). Thus the restriction of $H(Q,P)$ to the center manifold is positive definite and the level sets of H near the restpoint are $(4n-7)$-dimensional spheres. Since W^c is locally invariant and since the level sets of H keep orbits near the restpoint, there are invariant spheres for the n-body problem. Moreover, they are normally hyperbolic because the other $(2n-4)$ eigenvalues are hyperbolic. Hence every point in one of the spheres has an $(n-2)$-dimensional stable and unstable manifolds.



We can even say something about the flows in these spheres. The flow on W^c is a Hamiltonian system with a restpoint which is a non-degenerate minimum of the Hamiltonian. A theorem of Weinstein shows that there are $(2n-4)$ families of periodic solutions (parametrized by energy) emanating from the restpoint. Thus inside each sphere there are $(2n-4)$ periodic orbits representing small oscillations away from the collinear RE.

For example, when $n=3$, the spheres have dimension 5 inside the 8-dimensional manifold $S_{\omega_0'}$. The full energy manifolds (not just their intersections with W^c) have dimension 7 and the stable and unstable manifolds of points in the 5-spheres have dimension 1.

There are 3 periodic orbits in each nearby 5-sphere. One of these corresponds to the pair $\pm ik$ and is just the family of elliptical periodic solutions corresponding to the same RE. The other two are a planar and non-planar (that is, not in $\mathbb{R}^2$) small oscillation. When two of the masses are equal the nonplanar one looks like the figure at left: the bodies remain in a vertical plane and form an isosceles triangle



If we consider just the planar problem, we have 5-dimensional energy manifolds containing invariant 3-spheres. The union of the stable and unstable manifolds of all points of the sphere is 4-dimensional, that is, codimension-1 in the energy level. Thus these manifolds form separatrices in the energy level. In this way the collinear RE's take on a far greater significance than they have by virtue of being simple, explicit solutions

Next consider the Lagrangian, equilateral solutions of the $n=3$ body problem. We will normalize the size of the triangle so that, except for a translation to put the center of mass at the origin,

$$Q_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad Q_2' = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} \qquad Q_3' = \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} \qquad (Q = Q' - \overline{Q}')$$

The mutual distances are $r_{ij} = \sqrt{3}$ and the frequency of rotation turns out to be

$$k^2 = \frac{U(Q)}{Q^T M Q} = \frac{\overline{m}}{3\sqrt{3}} \qquad \overline{m} = m_1 + m_2 + m_3.$$

The matrix $M^{-1}B$ connected with non-planar stability is

$$M^{-1}B = \frac{1}{3\sqrt{3}} \begin{bmatrix} -(m_2+m_3) & m_2 & m_3 \\ m_1 & -(m_1+m_3) & m_3 \\ m_1 & m_2 & -(m_1+m_2) \end{bmatrix}$$

with eigenvalues $\mu = 0, -k^2, -k^2$. These give rise to the eigenvalues $\lambda = 0, 0, \pm ik, \pm ik$ for L as discussed above.

Analysis of planar stability involves the 12×12 matrix $\left[\begin{array}{c|c} \mathbf{K} & M^{-1} \\ \hline D' & K \end{array}\right]$, where D' is the planar second derivative matrix:

$$D_{ij}' = \frac{m_i m_j}{r_{ij}^3}\left[I - 3u_{ij}u_{ij}^T\right] \qquad D_{ii}' = -\sum_{j \neq i} D_{ij} \qquad 2\times 2 \text{ blocks.}$$

$$u_{ij} = \frac{Q_i - Q_j}{r_{ij}} \qquad \text{viewing } Q_i \in \mathbb{R}^2.$$

We can just as well use Q_i' for computing this. One finds

$$M^{-1}D' = \frac{k^2}{4\bar{m}} \begin{bmatrix} 5(m_2+m_3) & 3\sqrt{3}(m_3-m_2) & -5m_2 & 3\sqrt{3}\,m_2 & -5m_3 & -3\sqrt{3}m_3 \\ 3\sqrt{3}(m_3-m_2) & -(m_2+m_3) & 3\sqrt{3}m_2 & m_2 & -3\sqrt{3}m_3 & m_3 \\ -5m_1 & 3\sqrt{3}m_1 & 5m_1-4m_3 & -3\sqrt{3}m_1 & 4m_3 & 0 \\ 3\sqrt{3}m_1 & m_1 & -3\sqrt{3}m_1 & -m_1+8m_3 & 0 & -8m_3 \\ -5m_1 & -3\sqrt{3}m_1 & 4m_2 & 0 & 5m_1-4m_2 & 3\sqrt{3}m_1 \\ -3\sqrt{3}m_1 & m_1 & 0 & -8m_2 & 3\sqrt{3}\,m_1 & -m_1+8m_2 \end{bmatrix}.$$

Fortunately, we already know a lot about this 6×6 matrix. Namely we have the eigenvectors

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} \\ Q \\ \\ \end{bmatrix}, \quad u_4 = \begin{bmatrix} \\ K_3 Q \\ \\ \end{bmatrix},$$

$$\mu_1 = 0 \qquad \mu_2 = 0 \qquad \mu_3 = 2\frac{U(Q)}{Q'MQ} = 2k^2 \qquad \mu_4 = -k^2$$

where K_3 is the 6×6 matrix with blocks $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Let u_5, u_6, μ_5, μ_6 denote the other two eigenvectors and eigenvalues. Now $M^{-1}D'$ is symmetric with respect to the inner product $\langle Q, Q \rangle = Q^TMQ$ so its eigenvectors are M orthogonal. Hence u_5 and u_6 span the orthogonal complement of the span of $u_1, \cdots, u_4$. Now K_3 is an isometry of this metric and $u_2 = K_3 u_1$, $u_4 = K_3 u_3$ so K_3 leaves the span of $u_1 \cdots u_4$ invariant. Hence it must do the same for the plane spanned by u_5 and u_6. Since $K_3 u_5$ is orthogonal to u_5, it must be a multiple of u_6. Thus we may assume $u_6 = K_3 u_5$. From this it follows that the 4 vectors

$$\begin{bmatrix} u_5 \\ \hline 0 \end{bmatrix} \quad \begin{bmatrix} u_6 \\ \hline 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \hline Mu_5 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \hline Mu_6 \end{bmatrix}$$

span an invariant subspace for L (recall $\mathbb{K} = k K_3$). The matrix of L_1 on this space is

$$\left[\begin{array}{cc|cc} 0 & -k & 1 & 0 \\ k & 0 & 0 & 1 \\ \hline \mu_5 & 0 & 0 & -k \\ 0 & \mu_6 & k & 0 \end{array}\right]$$

and by our lemma, the characteristic polynomial is

$$\lambda^4 + (2-\alpha_5-\alpha_6)k^2\lambda^2 + (1+\alpha_5)(1+\alpha_6)k^4$$

where $\alpha_5 = \frac{\mu_5}{k^2}$, $\alpha_6 = \frac{\mu_6}{k^2}$. Thus we just need to find the eigenvalues μ_5 and μ_6.

We can find $\mu_5+\mu_6$ from the trace:

$$\text{trace } M^{-1}D' = 2k^2 = \sum_{i=1}^{6} \mu_i = k^2 + \mu_5 + \mu_6 \quad \Longrightarrow \mu_5 + \mu_6 = k^2$$

or $\alpha_5 + \alpha_6 = 1$

Now

$$\sum_{i=1}^{6} \mu_i^2 = tr\left[(M^{-1}D')^2\right] = \frac{k^4}{2\bar{m}^2}\left(20(m_1^2+m_2^2+m_3^2)+13(m_1m_2+m_1m_3+m_2m_3)\right)$$

To find the characteristic polynomial we need to find

$$(1+\alpha_5)(1+\alpha_6)k^4 = (1+\alpha_5+\alpha_6+\alpha_5\alpha_6)k^4 = (2+\alpha_5\alpha_6)k^4.$$

Now

$$\sum_{i<j} \mu_i\mu_j = k^4\sum_{i<j}\alpha_i\alpha_j = \frac{1}{2}\left(tr[M^{-1}D]\right)^2 - \frac{1}{2}tr\left[(M^{-1}D)^2\right]$$

$$= \frac{k^4}{4\bar{m}^2}\left(-12(m_1^2+m_2^2+m_3^2)+3(m_1m_2+m_1m_3+m_2m_3)\right)$$

and

$$\sum_{i<j}\alpha_i\alpha_j = \alpha_3(\alpha_4+\alpha_5+\alpha_6)+\alpha_4(\alpha_5+\alpha_6)+\alpha_5\alpha_6$$

$= 0$ [illegible] $1+\alpha_5\alpha_6$

Hence

$$(2+\alpha_5\alpha_6)k^4 = 3k^4 + \frac{k^4}{4\bar{m}^2}\left(\sim\!\sim\right)$$

$$= \frac{27(m_1m_2+m_1m_3+m_2m_3)}{4\bar{m}^2}k^4$$

If we set $\lambda = kz$, the characteristic polynomial is

$$z^4 + z^2 + \frac{27(m_1m_2+m_1m_3+m_2m_3)}{4\bar{m}^2} = 0$$

This has imaginary roots iff its discriminant is positive, that is

$$27(m_1m_2+m_1m_3+m_2m_3) < \bar{m}^2.$$

Otherwise it has 4 complex, non-imaginary roots. This is the stability criterion of Routh.

The inequality holds only if one mass is much larger than the other 2.

If we normalize so that $\bar{m} = 1$ by setting $m_1 = \alpha, m_2 = \beta, m_3 = 1-\alpha-\beta$ we find

$$(\alpha+\beta)(1-(\alpha+\beta)) + \alpha\beta < \frac{1}{27}$$

and so certainly

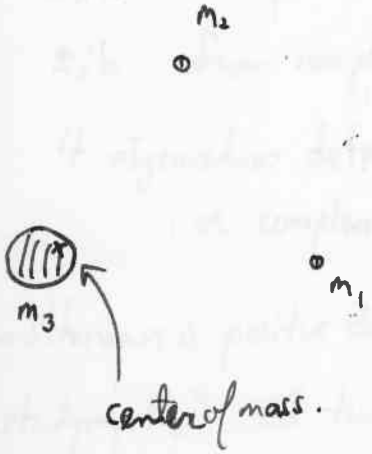
$$(\alpha+\beta)(1-(\alpha+\beta)) < \frac{1}{27}.$$

This holds only if

$$\alpha+\beta < \tfrac{1}{2}\left(1-\tfrac{\sqrt{69}}{9}\right) \simeq 0.03852$$

(or $\alpha+\beta > \frac{1}{2}(1+\frac{\sqrt{69}}{2}) \simeq .9614\cdots$, but assume m_3 was the largest mass)

The resulting periodic solution looks like two tiny planets in orbit around a large sun.



Note: The equilateral RE is always a minimum of the potential $U|_s$ (Morse index 0). Nevertheless its stability changes as the masses change. Thus Morse index does not uniquely determine stability type.

Consider a neighborhood of the Lagrangian restpoints in the $(6n-10) = 8$ dimensional quotient space. The eigenvalues will be the 18 eigenvalues in $\mathbb{R}^{6n=18}$ with the exceptions:

$0, 0, \pm ik, \pm ik$ from E_1 (center of mass)
(first 4 planar, last 2 non-planar)

$0, 0, \pm ik$ from E_2 (leaving two copies of $\pm ik$)
(first 2 planar, last 2 non-planar)

The 8 remaining are

$\pm ik$ from planar block of E_2
$\pm ik$ from nonplanar block of E_2
4 eigenvalues determined above which may be all imaginary or complex, non-imaginary. These are planar.

The Hamiltonian is positive definite on the $\pm ik$ eigenspace (as we saw when studying E_2). It turns out to be indefinite on the remaining eigenspace. Thus, in the linearly stable case, we cannot use the Hamiltonian to get stability.

In the hyperbolic case, the $\pm ik$ eigenspace is tangent to the center manifold and we obtain normally hyperbolic invariant 3-spheres (as before we found 5-spheres). Restricting to the planar problem we have a normally hyperbolic 1-sphere. This is just the elliptic RE periodic orbit, which is therefore a hyperbolic periodic orbit.

In the linearly stable case, one expects 4 families of periodic orbits near the restpoint (on nearby energy levels). As long as there are no integer ratios of frequencies, this follows from the Lyapunov center theorem:

> If $\pm i\alpha$ are eigenvalues and no other eigenvalues are of the form $\pm i n\alpha$, $n \in \mathbb{Z}$, then there is a surface of periodic orbits tangent to the $\pm i\alpha$ eigenspace. The orbits are parametrized by energy.

Of course there <u>is</u> an integer resonance here but one can apply this to the other imaginary pairs.

Also, one expects that KAM theory will apply to give nearly quasi-periodic orbits, but to my knowledge, the necessary hypotheses have not been checked.

Rings We will study a limiting case of the n-body problem where all but one of the masses is small. Actually, it is convenient to suppose there are (n+1) masses and to work in the plane.

$$q = \begin{bmatrix} q_0 \\ \vdots \\ q_n \end{bmatrix} \in \mathbb{R}^{2n+2} \qquad \begin{array}{l} m_0 = 1 \\ m_i = \varepsilon \mu_i \quad , \mu_i > 0 \\ \varepsilon \text{ small parameter.} \end{array}$$

In problems like this it is not so good to use the metric $q^T M q$ to normalize the size. Another convenient choice of normalization is to look for RE with angular velocity $k = 1$. These solve

$$M^{-1} \nabla U(q) + \underset{\lambda = 1}{\uparrow} q = 0 \qquad *$$

We will suppose we have a family of solutions q^ε for $\varepsilon \to 0$ and also that the family converges to some limiting configuration $\bar{q} \notin \Delta$. The problem is to characterize such limits $\bar{q}$ and to study the stability of the relative equilibria q^ε, ε small. The first part was studied by G.R. Hall who called $\bar{q}$ a RE of the (1+n) body problem, that is, 1 big mass, n infinitesimal ones.

Using the center of mass equation gives (suppressing the ε superscript)

$$q_0 = -\varepsilon \sum_{j=1}^{n} \mu_j q_j = O(\varepsilon) \quad , \ \bar{q}_0 = 0$$

that is, the big mass ends up at the origin.

Remarkably, all the other masses must converge to the unit circle. To see this, note that the ith component of * is

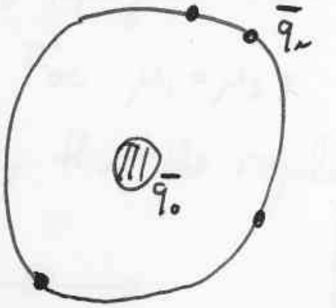
$$\varepsilon \sum_{j=1}^{n} \frac{\mu_j (q_j - q_i)}{r_{ij}^{3}} + \frac{(q_0 - q_i)}{r_{0i}^{3}} + q_i = 0$$

which gives

$$q_i \left(1 - \frac{1}{r_{i0}^{3}}\right) = \mathcal{O}(\varepsilon)$$

Since $q_i \to \bar{q}_i \neq 0$ (the limit $\bar{q} \notin \Delta$ so $\bar{q}_i \neq \bar{q}_j \ \forall i \neq j$), this gives

$$r_{i0} = 1 + \mathcal{O}(\varepsilon) \qquad \text{and} \qquad |q_i| = 1 + \mathcal{O}(\varepsilon).$$



Thus the limit has the structure of a "ring". It remains to see how the $\bar{q}_i$ are distributed around the ring. Write

$$\bar{q}_i = \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix}$$

and consider the inner product of the ith component of * with $\bar{q}_i^{\perp} = \begin{bmatrix} -\sin\theta_i \\ \cos\theta_i \end{bmatrix}$. Using $q_i \cdot q_i^{\perp} = 0$ and the center of mass equation and cancelling a factor of ε gives

$$\sum_{j \neq i} \mu_j \, q_j \cdot q_i^{\perp} \left[\frac{1}{r_{ij}^{3}} - 1 \right] = \mathcal{O}(\varepsilon)$$

In the limit we have $q_j \cdot q_i^{\perp} = -\cos\theta_j \sin\theta_i + \sin\theta_j \cos\theta_i = \sin(\theta_j - \theta_i)$ and $r_{ij}^2 = 2(1-\cos(\theta_i-\theta_j))$. Then the limiting equation is

$$\mu_i^{-1}\,\nabla_i\,\bar{V}(\theta_1,\cdots,\theta_n) = 0$$

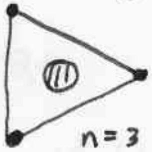
where

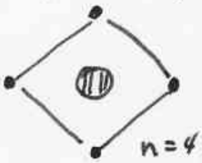
$$\bar{V}(\theta_1,\cdots,\theta_n) = \sum_{i<j}\frac{\mu_i\mu_j}{r_{ij}} + \frac{1}{2}\sum_{i<j}\mu_i\mu_j r_{ij}^2 .$$

Thus the vector $\Theta = (\theta_1,\cdots,\theta_n) \in T^n$ corresponding to the distribution of the small masses around the ring is a critical point of this potential function $\bar{V}(\theta)$.

As for the collinear problem, it is easy to show that $\bar{V}(\theta)$ must have a minimum on each component of $T^n \setminus \Delta$. There is no topological reason why more than one critical point should exist. For $\mu_1 = \mu_2 = \cdots = \mu_n$ one can show from symmetry that the regular n-gon is a critical point

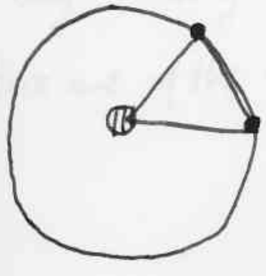

n=2


n=3


n=4

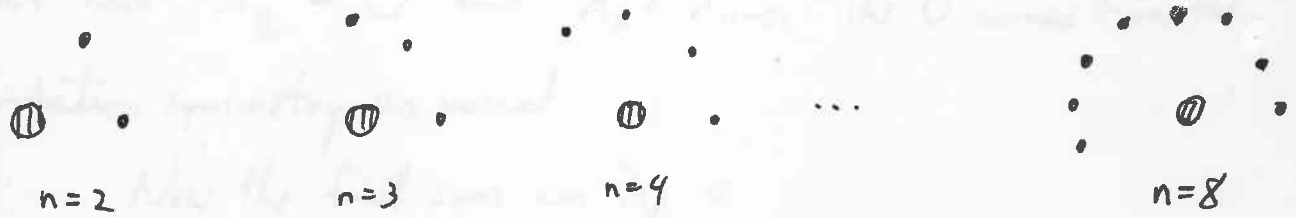
...

However, for n=2 we also have the equilateral triangle.



For non-equal masses, it is not at all clear what to expect.

We will now show that the regular n-gon is a minimum of $V(\theta)$ (with equal μ_i) if and only if $n \geq 7$. This implies that for $n \leq 6$ there must be another distribution of masses giving the minimum. Searching numerically, one finds an interesting sequence of irregular rings.



For $n = 7, 8$ there are two minima. For $n \geq 9$ no irregular minima were found. Hall proved that for n sufficient large (and μ_i equal) the regular n-gon is the only critical point.

For the proof of minimality of the n-gon we need to look at the Hessian $D^2V(\theta_1, \cdots, \theta_n)$. Taking $\mu_i = 1$ we find

$$V_{\theta_i\theta_j} = -\frac{3}{2r_{ij}^3} - \frac{\cos(\theta_i-\theta_j)}{2r_{ij}^3} - \cos(\theta_i - \theta_j)$$

$$V_{\theta_i\theta_i} = -\sum_{j\neq i} V_{\theta_i\theta_j}$$

This is a circulant matrix, that is, $V_{\theta_{i+k}\,\theta_{j+k}} = V_{\theta_i\theta_j}$ for any k using arithmetic mod n. The eigenvectors of any such matrix are of the form:

$$v = \begin{bmatrix} \rho \\ \rho^2 \\ \vdots \\ \rho^n = 1 \end{bmatrix} \qquad \rho = e^{2\pi i \, l/n} \text{ a root of unity}$$

Using this one gets the eigenvalues (from the last row of $D^2V(\theta)\cdot v$)

$$\lambda_\ell = \sum_{j=1}^{n-1} c_j(1-c_{j\ell}) + \frac{1}{2}\sum_{j=1}^{n-1}\frac{c_j(1-\ c_{j\ell})}{r_j^3} + \frac{3}{2}\sum_{j=1}^{n-1}\frac{(1-c_{j\ell})}{r_j^3} \qquad \ell=1,\cdots,n$$

$$\text{where } c_{j\ell} = \cos\left(\frac{2\pi j\ell}{n}\right),\ c_j = \cos\left(\frac{2\pi j}{n}\right),\ r_j^2 = 2(1-c_j).$$

We have $\lambda_n = 0$ and $\lambda_\ell = \lambda_{n-\ell}$. The 0 comes from the rotation symmetry as usual.

Now the first sum in λ_ℓ is

$$\sum_{j=1}^{n-1} c_j(1-c_{j\ell}) = \sum_{j=1}^{\overset{\swarrow}{n}} c_j(1-c_{j\ell}) = -\sum_{j=1}^{n} c_j c_{j\ell} = \begin{cases} 0 & j\neq 1,n-1 \\ -\frac{n}{2} & j=1,n-1\end{cases}$$

The other two sums can be combined as

$$\frac{1}{2}\sum_{j=1}^{n-1}\frac{(3+c_j)(1-c_{j\ell})}{r_j^3} > 0$$

Thus for $\ell \neq 1, n-1$ we have $\lambda_\ell > 0$. To be a minimum we also need $\lambda_1 = \lambda_{n-1} > 0$. Now

$$\lambda_1 = \frac{1}{2}\sum_{j=1}^{n-1}\frac{(3+c_j)(1-c_{j\ell})}{r_j^3} - \frac{n}{2} \underset{\substack{\uparrow \\ r_j^2=2(1-c_j)}}{=} \frac{1}{4}\sum_{j=1}^{n-1}\frac{(3+c_j)}{r_j} - \frac{n}{2}$$

Writing $3+c_j = 4 - \frac{1}{2}r_j^2$ gives

$$\lambda_1 = \sum_{j=1}^{n-1}\frac{1}{r_j} - \frac{1}{8}\sum_{j=1}^{n-1} r_j - \frac{n}{2} = \sum_{j=1}^{n-1}\frac{1}{r_j} - \frac{1}{4}\cot\frac{\pi}{2n} - \frac{n}{2}$$

For a minimum we need

$$A_n = \frac{1}{n}\sum_{j=1}^{n-1} \frac{1}{r_j} > \frac{1}{4n}\cot\frac{\pi}{2n} + \frac{1}{2}$$

The sum A_n is a geometrical quantity associated to an n-gon. It diverges like log(n) while the right side remains bounded so the inequality will clearly be true for n suff. large. Direct evaluation of both sides for small n gives:

n	A_n	$\frac{1}{4n}\cot\frac{\pi}{2n}+\frac{1}{2}$
2	0.25	0.625
3	0.3849	0.644
4	0.478	0.651
5	0.551	0.654
6	0.609	0.655 ←
7	0.65804	0.65647
8	0.701	0.657

Some further estimates prove that $A_n > \frac{1}{4n}\cot\frac{\pi}{2n} + \frac{1}{2}$ for $n \geq 8$. But we will skip these.

We have been considering the limit of a convergent sequence q^ε of RE. Suppose now that we are given $\bar{q}$, a critical point of $V(\theta)$. If $\bar{q}$ is a non-degenerate critical point (nullity $D^2V = 1$, the min. allowed by symmetry) then one can use the implicit function theorem to show that there is indeed a sequence of RE, q^ε, with $q^\varepsilon \to \bar{q}$.

Finally we turn to the question of linear stability. Given a sequence $q^\varepsilon \to \bar{q}$ as above, what can be said about linear stability of the resulting circular periodic orbit. This was studied by Maxwell for the case of the regular n-gon with equal μ_i. He was interested in this as a model of a single ring of Saturn. He found that the ring is linearly stable for ε sufficiently small.

We will show:

If $\bar{q}$ is a non-degenerate critical point of $V(\theta)$ then q^ε is linearly stable for ε suff small if and only if $\bar{q}$ is a local minimum of $V(\theta)$.

This implies Maxwell's result for $n \geq 7$. In fact, the ring is unstable for $n \leq 6$. For $n \leq 8$ the irregular rings found numerically are linearly stable as well.

The proof involves the usual matrix:

$$L = \begin{array}{cl} \left[\begin{array}{c|c} K & M^{-1} \\ \hline D^2U(q) & K \end{array}\right] & \begin{array}{l} 2n+2 \\ 2n+2 \end{array} \\ \begin{array}{cc} 2n+2 & 2n+2 \end{array} & \end{array} \qquad q = q^\varepsilon.$$

If $v = \left[\begin{array}{c} v_1 \\ \hline v_2 \end{array}\right]\begin{array}{l} 2n+2 \\ 2n+2 \end{array}$ is an eigenvector with eigenvalue λ then

$$v_2 = M(K+\lambda I)v_1$$

$$Cv_1 = 0 \quad \text{where} \quad C = M^{-1}D^2U(q) - (K+\lambda I)^2$$

For vectors v_1 tangent to the set of configurations with $\bar{y}=0$ we have

$$v_{10} + \varepsilon \sum_{j=1}^{n} \mu_j v_{1j} = 0$$

One can use this to eliminate v_{10} and reduce to a $2n \times 2n$ matrix $\hat{C}$ instead of C_1. Motivated by the limiting shape of the configuration we will split up v_{1j} into radial and angular parts

$$v_{1j} = \rho_j \begin{bmatrix} \cos\theta_j \\ \sin\theta_j \end{bmatrix} + J_j \begin{bmatrix} -\sin\theta_j \\ \cos\theta_j \end{bmatrix}.$$

In this basis one finds (after a while) that

$$\hat{C} = \begin{array}{c} \left[\begin{array}{c|c} (3-\lambda^2)I + O(\varepsilon) & -2\lambda I + O(\varepsilon) \\ \hline 2\lambda I + O(\varepsilon) & -\lambda^2 I + \varepsilon\mu^{-1}D^2V(\theta) + O(\varepsilon^2) \end{array}\right] \begin{array}{l} n \\ n \end{array} \\ \begin{array}{cc} n \qquad\qquad & \qquad\qquad n \end{array} \end{array}$$

where $\mu = \mathrm{diag}[\mu_1, \cdots, \mu_n]$. The characteristic polynomial is

$$P(\lambda) = \lambda^{2n}(1+\lambda^2)^n + O(\varepsilon)$$

so the eigenvalues tend to $\pm i$ or 0. Setting $\lambda = \sqrt{\varepsilon}\,h$ gives

$$P(\sqrt{\varepsilon}\,h) = \varepsilon^n \det\left(3\mu^{-1}D^2V(\theta) + h^2 I\right) + O(\varepsilon^{n+1})$$

It follows that the eigenvalues which tend to 0 are of the form

$$\lambda = \pm\sqrt{\varepsilon}\,\sqrt{-\gamma} \quad \text{where } \gamma \text{ is an eigenvalue of}$$

$3\mu^{-1}D^2V(\theta)$. If any of these eigenvalues are negative then we get real λ's and the RE is unstable.

If $\bar{q}$ is a local minimum, all γ (except the single 0) are positive and we have a chance for stability. We will prove strong stability by using the $\omega(v,\bar{v}) \neq 0$ test.

Assume that $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigenvector. Then we have

$v_2 = M(K+\lambda I)v_1$ and so (if $\lambda = \alpha + i\beta$)

$$\omega(v,\bar{v}) = v_1^T\bar{v}_2 - v_2^T\bar{v}_1 = 2i\beta\, v_1^T M \bar{v}_1 - 2v_1^T M K \bar{v}_1$$

Writing v_{1i} in terms of ρ_i + ζ_i as above and eliminating v_{10} using the center of mass gives:

$$\omega(v,\bar{v}) = \varepsilon\left[2i\beta(\rho^T\mu\bar{\rho} + \zeta^T\mu\bar{\zeta}) - 2\bar{\zeta}^T\mu\bar{\rho} + 2\rho^T\mu\bar{\zeta}\right] + \overset{\substack{\text{from } v_{10} \\ \downarrow}}{\mathcal{O}}(\varepsilon^2)$$

Now we can normalize the eigenvector so $\rho^T\mu\bar{\rho} + \zeta^T\mu\bar{\zeta} = 1$. Then we need to show

$$\left[2i\beta - 2\zeta^T\mu\bar{\rho} + 2\rho^T\mu\bar{\zeta}\right] \neq 0$$

for ε suff small.

For the eigenvalues of the form $\lambda = \pm i + \mathcal{O}(\varepsilon)$, the eigenvectors satisfy $\zeta = \pm 2i\rho + \mathcal{O}(\varepsilon)$ and we get $\rho^T\mu\bar{\rho} + \zeta^T\mu\bar{\zeta} = 5\rho^T\mu\bar{\rho} = 1$ and

$$\omega(v,\bar{v}) = \varepsilon\left[\pm \tfrac{2}{5} i\right] + \mathcal{O}(\varepsilon^2)$$

which is non-zero for ε suff small.

Next consider the eigenvalues of the form $\lambda = \sqrt{\varepsilon}\,\zeta$. We have $\zeta^2 \to -\gamma$ where γ is an eigenvalue of $3\mu^{-1}D^2V(\theta)$. We are assuming $\gamma > 0$ so $\lambda = \pm i\sqrt{\varepsilon\gamma} + o(1)$ and so $z = \pm\sqrt{\varepsilon\gamma} + o(1)$.

The eigenvectors satisfy

$$\xi = \pm \frac{2i}{3}\sqrt{\varepsilon\gamma}\; z + \cdots$$

which gives $z^T \mu \bar{z} = 1 + \cdots$

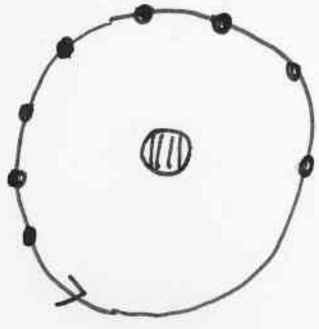
and

$$\omega(v, \bar{v}) = \varepsilon\left[\mp \frac{2i}{3}\sqrt{\varepsilon\gamma} + \cdots\right]$$

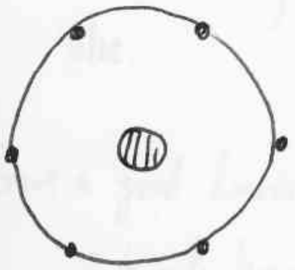
which is also nonzero for ε suff small.

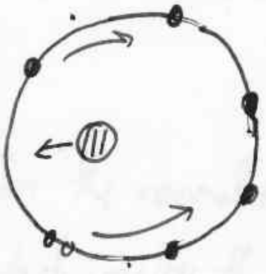
These computations show that the Hamiltonian sign of the $\pm i$ eigenvalues is $+$ while that of the $\sqrt{\varepsilon}\,\zeta$ eigenvalues is $-$.

Thus Lagranges equilateral periodic solution is only the first of a sequence of linearly stable periodic orbits in the form of irregular rings.



With a little more effort one can analyze the nature of the instability of the regular n-gon, $n \leq 6$. It is associated to the eigenvalue λ_1^{-} of the circulant matrix. The corresponding eigenvectors of L represent a motion in which the central mass moves away from O while the ring particles move the other way





This grows exponentially leading to instability.

Open Problems We will close by describing some possible areas for future research.

1. Find geometrically interesting attractors for the gradient flow $\tilde{\nabla} U|_S$.

   In this way one could generalize the situation for the collinear configurations in S : Knowing C is an attractor implies existence of a minimum somewhere else.

2. Give a good lower bound for the normal index of a planar CC.

   We only know ≥ 1 which implies that a planar CC is not the minimum of $U|_S$. A better estimate would make 3D Morse theory much more powerful.

3. Give an upper bound for the index of a non-planar CC.

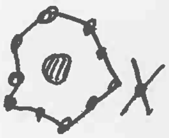
   Analogous to Palmore's $\leq n-2$ for the planar index of a planar CC. The natural guess is $2n-4$ but this has not been proved

4. Show that for almost all choices of $(m_1, \cdots, m_n)$ there are only finitely many CC's (and they are non-degenerate).

5. Show that every linearly stable RE has a dominant mass, that is, some m_i is much larger than all the others.

   All known examples are of this type. A counterexample would be even better

6. Show that every linearly stable RE is ring-like, that is, the small masses are arranged near a circle around the big mass and the segments connecting them are nearly tangent to the circle.



Again, all known examples are of this type.

7. Do KAM near RE.

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(The articles are ones having something to do with the topics covered in the lectures, plus a few favorites)

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