Matrix Integrals and Map Enumeration: An Accessible Introduction

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This paper is dedicated to the memory of Claude Itzykson

Abstract—Physicists working in two-dimensional quantum gravity invented a new method of map enumeration based on computation of Gaussian integrals over the space of Hermitian matrices. This paper explains the basic facts of the method and provides an accessible introduction to the subject.

Keywords—Maps, Imbedded graphs, Enumerative combinatorics, Matrix integrals, Quantum field theory, String theory.

1. INTRODUCTION

Map enumeration is a vast and well developed branch of enumerative combinatorics. It is impossible to give an account of this theory in one paper. The techniques of computation of matrix integrals is also an elaborate subject full of ingenious methods and technically difficult results. This paper is not a survey of these methods. Our goal is much more modest: we will try to explain the word *and* in the title. That is, why and how are matrix integrals related to map enumeration?

The last decades witnessed a spectacular development of various interrelations between combinatorial maps (or "imbedded graphs") and many other branches of mathematics and theoretical physics. Grothendieck was first to note that the absolute Galois group acts on maps, and this remark led to a development of the theory of "dessins d'enfants" [1] which further explores the connections between combinatorics of maps and Galois theory. Maps became relevant to the knot theory, via Vassiliev knot invariants [2]. New relations of maps with groups were found [3], and so on.

Among all these exciting new developments one of the most interesting is the fact that physicists (working in two-dimensional quantum gravity) became interested in map enumeration. This is how very classic problems of the enumerative combinatorics suddenly became a hot point in theoretical physics. But, as is often the case with physicists, they developed their own, highly original techniques of map enumeration based on the computation of integrals over the space of

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Hermitian matrices. The foundations of the method were developed in [4,5]. In physical literature they also cite [6] as a source of the main idea.

Unfortunately, these extremely interesting methods are still largely unknown to combinatorial community. The major reason is the difference of language and background, the different set of notions and techniques supposed to be "known to everybody". The goal of the present paper is to build a bridge between two theories and to explain why and how map enumeration is related to matrix integrals. We claim no originality, and we do not touch here the difficult (and most substantial) parts of the theory, namely, methods of computation of integrals. We explain here only the simplest part, i.e., the connection itself. (This part is presumed to be so simple that the relevant explanations were often omitted, thus making the reading of original papers very difficult for a noninitiated reader.)

Our bibliography is not complete. For the latest developments we refer the reader to the survey [7]. For subsequent reading we heartily recommend [8]. The comprehensive account of all the aspects mentioned above (dessins d'enfants, group theory, matrix integrals) may be found in [9].

Reading physical literature is sometimes a very difficult task, but it can be enormously instructive. We hope to share with the reader our joy of a journey across this new world.

2. GAUSSIAN MEASURES AND INTEGRALS

The aim of this section is to introduce Gaussian measures in Euclidean spaces, and to develop techniques of computing the integrals with respect to these measures.

2.1. Basic Notions

The material of this section is very standard and may be found in many textbooks of probability theory; see, for example, [10] . For an exposition closer to physics (including Wick formula) see [11].

2.1.1. Standard Gaussian measure on the line

This is the measure μ on the real line $\mathbb{R}$, with the density

$$d\mu(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx, \qquad x \in \mathbb{R}.$$

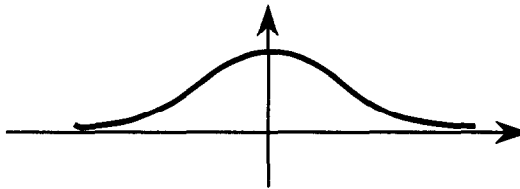You may see the graph of the density in the following figure.

Figure 1. Standard Gaussian density on the line.

This measure has the following properties.

(1) It is *normalized*, that is, the integral of the density over the line is equal to 1:

$$\int_{\mathbb{R}} d\mu(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = 1.$$

(2) The mean is equal to 0:

$$\int_{\mathbb{R}} x \, d\mu(x) = 0.$$

(3) The variance is equal to 1:

$$\int_{\mathbb{R}} x^2 \, d\mu(x) = 1.$$

(4) The characteristic function (or Fourier transform) is

$$\varphi(t) := \int_{\mathbb{R}} e^{itx} \, d\mu(x) = e^{-t^2/2}.$$

NOTATION 2.1. We will regularly use the following notation borrowed from physics: for any measure μ on X, and for any function $f : X \to \mathbb{R}$ or $f : X \to \mathbb{C}$, we will denote by $\langle f \rangle$ the *mean*, or *average* of f with respect to the measure μ:

$$\langle f \rangle := \int_X f(x) \, d\mu(x).$$

The measure μ and the space X that are not explicitly mentioned in this notation will usually be clear from the context. For example, the above formulae may be rewritten as

$$\langle 1 \rangle = 1, \qquad \langle x \rangle = 0, \qquad \langle x^2 \rangle = 1, \qquad \varphi(t) := \langle e^{itx} \rangle = e^{-t^2/2}.$$

2.1.2. General Gaussian measure on the line

This measure has the density

$$d\mu(x) := \frac{1}{\sqrt{2\pi}\sigma} e^{-((x-m)^2)/2\sigma^2} \, dx, \qquad x, m, \sigma \in \mathbb{R}, \quad \sigma > 0. \tag{1}$$

This measure is also normalized, and the mean, the variance, and the characteristic function are as follows:

$$\langle x \rangle = m, \qquad \langle (x - m)^2 \rangle = \sigma^2, \qquad \varphi(t) := \langle e^{itx} \rangle = e^{imt - (\sigma^2 t^2/2)}. \tag{2}$$

REMARK 2.2. As is well known, a probabilistic distribution is uniquely determined by its characteristic function. Hence, we could give the last formula for $\varphi(t)$ as a *definition* of a Gaussian measure on the line.

2.1.3. Gaussian measure in $\mathbb{R}^n$

This measure will be the main object of our study. To introduce it we must start from its characteristic function.

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. By (x, y) we denote the ordinary scalar product in $\mathbb{R}^n$, that is, $(x, y) := x_1 y_1 + \cdots + x_n y_n$.

DEFINITION 2.3. A measure μ on $\mathbb{R}^n$ is called *Gaussian*, if its characteristic function has the form

$$\varphi(t) := \int_{\mathbb{R}^n} e^{i(t,x)} \, d\mu(x) = \exp\left\{ i(m, t) - \frac{1}{2}(Ct, t) \right\},$$

where $t, m \in \mathbb{R}^n$ are vectors, and $C = (c_{ij})$ is a nonnegatively defined $(n \times n)$-matrix of quadratic form. Vector m is the *mean vector* of measure μ, that is, $\langle x_i \rangle = m_i$, and matrix C is its *covariance matrix*, that is, $\langle (x_i - m_i)(x_j - m_j) \rangle = c_{ij}$.

REMARK 2.4. In what follows, for the sake of simplicity, we consider only the case of the zero mean $m = 0$, thus

$$\varphi(t) = \exp\left\{ -\frac{1}{2}(Ct, t) \right\} \tag{3}$$

and

$$\langle x_i \rangle = 0, \quad \langle x_i x_j \rangle = c_{ij}, \qquad i,j = 1, 2, \ldots, n.$$

REMARK 2.5. The covariance matrix C may well be degenerate. Then the measure has no density and is concentrated on a vector subspace of $\mathbb{R}^n$. When C is *non*degenerate, the measure has density

$$d\mu(x) = \text{Const} \times \exp\left\{-\frac{1}{2}(Bx, x)\right\} dx, \tag{4}$$

where the matrix $B = C^{-1}$. For this measure to be normalized, i.e., the measure of the whole space $\mathbb{R}^n$ to be equal to 1, we should take the constant factor in the last formula equal to

$$\text{Const} = (2\pi)^{-n/2}(\det B)^{1/2}. \tag{5}$$

All the above formulae (3)–(5) are consistent with the one-dimensional case (1),(2), where $C = (\sigma^2)$, $B = (\sigma^{-2})$, and $\text{Const} = (2\pi)^{-1/2}\sigma^{-1}$.

A Gaussian measure in $\mathbb{R}^n$ is called *standard* if both B and C are identity matrices.

LEMMA 2.6. *Let μ be a Gaussian measure in $\mathbb{R}^n$, and let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear operator. Then A induces a measure ν in $\mathbb{R}^k$ which is also Gaussian.*

PROOF. Let C be the covariance matrix of μ, and let $x, t \in \mathbb{R}^n$, $y, s \in \mathbb{R}^k$, and $y = Ax$. Then

$$\varphi_\nu(s) := \left\langle e^{i(s,y)} \right\rangle = \left\langle e^{i(s,Ax)} \right\rangle = \left\langle e^{i(A^*s,x)} \right\rangle = \varphi_\mu(A^*s),$$

where $A^* : \mathbb{R}^k \to \mathbb{R}^n$ is the operator adjoint to A. What remains is to substitute A^*s for t in $\varphi_\mu(t) = \exp\{-(1/2)(Ct,t)\}$, which gives the covariance matrix of ν equal to ACA^*.

REMARK 2.7. In the above lemma, the dimension k may be less than n, equal to n, or greater than n, and the operator A may be of an arbitrary rank.

2.2. Integrals of Polynomials and Wick Formula

A great part of the machinery of quantum physics consists of methods of calculating the integrals with respect to a Gaussian measure (not necessarily in the finite dimensional case); see, for example, [11]. The goal of this section is to develop a technique for integration of polynomials.

2.2.1. Wick formula

Knowing that $\langle x_i \rangle = 0$, $i = 1, \ldots, n$ and $\langle x_i x_j \rangle = c_{ij}$, $i,j = 1, \ldots, n$, one can easily compute the integral of any polynomial of $x_1, \ldots, x_n$ of degree 2. What about higher degrees?

LEMMA 2.8. *If $f(x)$ is a monomial of odd degree, then $\langle f \rangle = 0$.*

The following theorem reduces the integration of any polynomial of an even degree to that of degree 2.

THEOREM 2.9. *(Wick formula.) Let $f_1, f_2, \ldots, f_{2k}$ be a set of linear functions (not necessarily different) of $x_1, \ldots, x_n$. Then,*

$$\langle f_1 f_2 \ldots f_{2k} \rangle = \sum \langle f_{p_1} f_{q_1} \rangle \langle f_{p_2} f_{q_2} \rangle \cdots \langle f_{p_k} f_{q_k} \rangle,$$

where the sum is taken over all the permutations $p_1 q_1 p_2 \ldots q_k$ of the set of indices $1, 2, \ldots, 2k$, such that $p_1 < p_2 < \cdots < p_k$, $p_1 < q_1, \ldots, p_k < q_k$.

The number of summands on the right-hand side is equal to

$$(2k-1)!! := 1 \times 3 \times 5 \times \cdots \times (2k-1).$$

A partition of the set $1, 2, \ldots, 2k$ into couples (p_i, q_i) satisfying the conditions of the Wick formula is called a *Wick coupling*.

EXAMPLE 2.10. As an example of application of the Wick formula let us compute a one-dimensional integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^4 e^{-x^2/2}\, dx = \langle x^4 \rangle.$$

We have $x^4 = f_1 f_2 f_3 f_4$, where $f_1 = f_2 = f_3 = f_4 = x$. Hence,

$$\langle f_1 f_2 f_3 f_4 \rangle = \langle f_1 f_2 \rangle \langle f_3 f_4 \rangle + \langle f_1 f_3 \rangle \langle f_2 f_4 \rangle + \langle f_1 f_4 \rangle \langle f_2 f_3 \rangle.$$

Now each $\langle f_i f_j \rangle = \langle x^2 \rangle = 1$; therefore the result is $1^2 + 1^2 + 1^2 = 3$.

In the same way, we may find $\langle x^{2k} \rangle = (2k - 1)!!$.

2.2.2. Wick formula: A sketch of the proof

In fact, the Wick formula is a very particular case of a formula well known in probability theory, which expresses the moments of a probability distribution in terms of its semi-invariants. The latter one, in its turn, is a "probabilistic interpretation" of the formula of logarithm of a power series. For more details see, for example [12], Chapter 2 "Semi-invariants and combinatorics".

We use the multi-index notation

$$\alpha := (\alpha_1, \ldots, \alpha_n), \qquad \alpha_i \in \mathbb{N},$$
$$|\alpha| := \alpha_1 + \cdots + \alpha_n,$$
$$\alpha! := \alpha_1! \times \cdots \times \alpha_n!,$$
$$t^\alpha := t_1^{\alpha_1} \ldots t_n^{\alpha_n}, \qquad \text{where} \quad t = (t_1, \ldots, t_n),$$
$$\frac{\partial^{|\alpha|} f}{\partial t^\alpha} := \frac{\partial^{|\alpha|} f}{\partial t_1^{\alpha_1} \ldots \partial t_n^{\alpha_n}}.$$

Let

$$f(t) = 1 + \sum_{|\alpha| > 0} \frac{m_\alpha}{\alpha!} t^\alpha$$

be the Taylor series for a function $f(t)$ satisfying $f(0) = 1$, where

$$m_\alpha = \left. \frac{\partial^{|\alpha|} f}{\partial t^\alpha} \right|_{t=0}.$$

Let the following be the Taylor series for $\log f(t)$:

$$\log f(t) = \sum_{|\alpha| > 0} \frac{s_\alpha}{\alpha!} t^\alpha.$$

The formula below expresses the coefficients m_α in terms of s_α.

PROPOSITION 2.11. *The Taylor coefficients m_α of a function f are expressed in terms of the Taylor coefficients s_α of $\log f$ by the following formula:*

$$m_\alpha = \sum_{k=1}^{|\alpha|} \frac{1}{k!} \sum_{\substack{\beta^1 + \cdots + \beta^k = \alpha, \\ |\beta^i| > 0}} \frac{\alpha!}{\beta^1! \ldots \beta^k!} \prod_{i=1}^{k} s_{\beta^i}.$$

Here β^i are also multi-indices; the interior sum is taken over all ordered k-tuples of multi-indices (but the factor $1/k!$ "kills the order").

REMARK 2.12. We skipped the condition of the analyticity of function f because the above formula is also valid for formal power series.

PROPOSITION 2.13. *Let μ be a probability distribution in $\mathbb{R}^n$ (an arbitrary one, not necessarily Gaussian), and*

$$\varphi(t) = \left\langle e^{i(t,x)} \right\rangle = \int_{\mathbb{R}^n} e^{i(t,x)} \, d\mu(x),$$

its characteristic function. If φ is analytic in a neighborhood of 0, then all the moments $m_\alpha := \langle x^\alpha \rangle$ exist, and

$$\varphi(t) = 1 + \sum_{|\alpha|>0} \frac{m_\alpha}{\alpha!} (it)^\alpha.$$

The coefficients s_α of the series

$$\log \varphi(t) = \sum_{|\alpha|>0} \frac{s_\alpha}{\alpha!} (it)^\alpha,$$

are called *semi-invariants* of the distribution μ, so the formula of Proposition 2.11 expresses the moments of a probability distribution in terms of its semi-invariants.

The formulae and computations involving semi-invariants are often much simpler than those involving moments. A Gaussian distribution is a striking example of this phenomenon, as for the characteristic function (3), its logarithm is no more an infinite series but a quadratic polynomial

$$\log \varphi(t) = -\frac{1}{2}(Ct, t),$$

so $s_\alpha \neq 0$ implies $|\alpha| = 2$, and for such an α we have $s_\alpha = m_\alpha = c_{ij}$ (here $\alpha_i = \alpha_j = 1$ if $i \neq j$, and $\alpha_i = 2$ if $i = j$, while all the other components of the multi-index α are equal to 0).

Return now to the Wick formula. Let $y_1 = f_1(x), \ldots, y_{2k} = f_{2k}(x)$. According to Lemma 2.6 vector $(y_1, \ldots, y_{2k})$ has a Gaussian distribution, and our goal is to compute, for this distribution, the moment $\langle y_1 \ldots y_{2k} \rangle$ which corresponds to the multi-index $\alpha = (1, 1, \ldots, 1)$. Now we may apply the formula of Proposition 2.11 and to split α into the sums of multi-indices β^i, of which survive only those of the form $(0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$, having only two nonzero entries.

The Wick formula is proved.

3. THE SPACE OF HERMITIAN MATRICES

In this section, we consider a very special case of the previous construction, namely, a Gaussian measure on the space of Hermitian matrices.

3.1. Definitions and Notation

Let $H = (h_{ij})$ denote an Hermitian $(N \times N)$-matrix; that is, its entries are complex numbers: $h_{ij} \in \mathbb{C}$ and $h_{ij} = \overline{h_{ji}}$, where the bar denotes the complex conjugation. We denote by $\mathcal{H}_N$ the space off all such matrices.

Every Hermitian matrix may be described by N^2 real parameters: $x_{ii} = h_{ii} \in \mathbb{R}$, $i = 1, \ldots, N$ for diagonal entries, and $x_{ij} = \text{Re}(h_{ij})$, $y_{ij} = \text{Im}(h_{ij})$, $1 \leq i < j \leq N$ for over-diagonal entries (the under-diagonal entries being determined by the over-diagonal ones via complex conjugation). Thus, the space $\mathcal{H}_N$ is isomorphic to the Euclidean space $\mathbb{R}^{N^2}$. We introduce the ordinary Lebesgue measure in $\mathcal{H}_N$:

$$dv(H) := \prod_{i=1}^{N} dx_{ii} \prod_{i<j} dx_{ij} \, dy_{ij}.$$

In principle, the space $\mathcal{H}_N \cong \mathbb{R}^{N^2}$ does not differ from any other Euclidean space of the same dimension. But some of the characteristics of the Gaussian measure on it will be expressed in

terms of matrix operations. In order to introduce a Gaussian density in the form (4), we need a nondegenerate quadratic form in $\mathcal{H}_N$. We take as such a form the following one: $\text{tr}(H^2)$.

Let us see how this form looks in the coordinates x_{ij}, y_{ij}. A term $(\cdot)_{ik}$ of H^2 is equal to $\sum_{j=1}^N h_{ij} h_{jk}$; hence, a diagonal term $(\cdot)_{ii}$ is equal to $\sum_{j=1}^N h_{ij} h_{ji}$, and hence, the trace of H^2 is equal to

$$\text{tr}\left(H^2\right) = \sum_{i,j=1}^N h_{ij} h_{ji} = \sum_{i,j=1}^N h_{ij}\overline{h_{ij}} = \sum_{i=1}^N x_{ii}^2 + \sum_{i\neq j}\left(x_{ij}^2 + y_{ij}^2\right) = \sum_{i=1}^N x_{ii}^2 + 2\sum_{i<j}\left(x_{ij}^2 + y_{ij}^2\right).$$

Let us specify matrix B of this quadratic form (see (4)). It must be of size $N^2 \times N^2$. But the above expression shows that this matrix is very simple; indeed, it is diagonal! We have N diagonal terms equal to 1 (they correspond to the coordinates x_{ii}), and $N^2 - N$ terms equal to 2 (they correspond to the coordinates x_{ij}, y_{ij} for $i < j$):

$$B = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 2 & & & & \\ & & & & 2 & & & \\ & & & & & \ddots & & \\ & & & & & & 2 \end{pmatrix}.$$

For this matrix, it is very easy to find all the necessary ingredients for the subsequent work. For example,

$$\det B = 2^{N^2 - N},$$

so the density (4),(5) takes the form

$$d\mu(H) = (2\pi)^{-N^2/2} 2^{(N^2-N)/2} \exp\left\{-\frac{1}{2}\text{tr}\left(H^2\right)\right\} dv(H). \tag{6}$$

EXERCISE 3.1. Compute the covariance matrix $C = B^{-1}$. Infer from it that $\langle x_{ii}^2 \rangle = 1$, and $\langle x_{ij}^2 \rangle = \langle y_{ij}^2 \rangle = 1/2$ for $i < j$.

We will also need the following preparatory lemma.

LEMMA 3.2. *Taking as linear functions of the coordinates x_{ij}, y_{ij} the matrix elements h_{ij}, we have*

$$\langle h_{ij} h_{ji} \rangle = 1,$$

while all the other second moments are equal to zero:

$$\langle h_{ij} h_{kl} \rangle = 0, \qquad \text{whenever } (i,j) \neq (l,k).$$

PROOF. If $i = j$, then

$$\langle h_{ii}^2 \rangle = \langle x_{ii}^2 \rangle = 1.$$

If $i \neq j$, then

$$\langle h_{ij} h_{ji} \rangle = \langle h_{ij}\overline{h_{ij}} \rangle = \langle x_{ij}^2 + y_{ij}^2 \rangle = \frac{1}{2} + \frac{1}{2} = 1.$$

All the other second moments of the type $\langle h_{ij} h_{kl} \rangle$, $(i,j) \neq (l,k)$ involve only off-diagonal terms of the covariance matrix C.

3.2. An Example

Now, we are ready to try the efficiency of the developed techniques for computing an integral of some polynomial. Let us consider the following one: $\mathrm{tr}(H^{2k})$. It is a polynomial of degree $2k$ of N^2 variables h_{ij}, which are, in their turn, linear functions of x_{ij}, y_{ij}. More precisely,

$$\mathrm{tr}\left(H^{2k}\right) = \sum h_{i_1 i_2} h_{i_2 i_3} \ldots h_{i_{2k-1} i_{2k}} h_{i_{2k} i_1}, \tag{7}$$

where the sum is taken over all N^{2k} combinations of indices $i_1, i_2, \ldots, i_{2k}$. Consider the integral

$$\left\langle \mathrm{tr}\left(H^{2k}\right)\right\rangle.$$

In order to compute it, we must apply the Wick formula to expression (7). Let us look at how it works in a particular case.

EXAMPLE 3.3. Let $k = 4$, so we deal with the sum of the N^8 products of the form

$$h_{i_1 i_2} h_{i_2 i_3} h_{i_3 i_4} h_{i_4 i_5} h_{i_5 i_6} h_{i_6 i_7} h_{i_7 i_8} h_{i_8 i_1}.$$

Let us choose an arbitrary Wick coupling (out of $7!! = 105$ ones). For example, let us couple $h_{i_1 i_2}$ with $h_{i_4 i_5}$; $h_{i_2 i_3}$ with $h_{i_5 i_6}$; $h_{i_3 i_4}$ with $h_{i_8 i_1}$; and $h_{i_6 i_7}$ with $h_{i_7 i_8}$; in other words, let us consider the product

$$\langle h_{i_1 i_2} h_{i_4 i_5}\rangle \langle h_{i_2 i_3} h_{i_5 i_6}\rangle \langle h_{i_3 i_4} h_{i_8 i_1}\rangle \langle h_{i_6 i_7} h_{i_7 i_8}\rangle. \tag{8}$$

Recalling Lemma 3.2, we see that each factor of this product has quite good chances to be equal to 0 (and then the whole product will become 0). In order for the product to be nonzero, we need *all* of its factors to be equal to 1 (then the product itself is also equal to 1). And for that, we must impose rather restrictive conditions on indices (see once more Lemma 3.2):

$$\langle h_{i_1 i_2} h_{i_4 i_5}\rangle = 1 \iff i_1 = i_5, \quad i_2 = i_4,$$
$$\langle h_{i_2 i_3} h_{i_5 i_6}\rangle = 1 \iff i_2 = i_6, \quad i_3 = i_5,$$
$$\langle h_{i_3 i_4} h_{i_8 i_1}\rangle = 1 \iff i_3 = i_1, \quad i_4 = i_8,$$
$$\langle h_{i_6 i_7} h_{i_7 i_8}\rangle = 1 \iff i_6 = i_8, \quad i_7 = i_7.$$

Finally, we have the following "chains of equalities":

$$i_1 = i_5 = i_3 = i_1,$$
$$i_2 = i_4 = i_8 = i_6 = i_2,$$
$$i_7 = i_7,$$

which gives us N^3 possible combinations of indices (the indices i_1, i_2, and i_7 may be chosen arbitrarily, while the values of all the other indices are determined by this choice). This fact is usually expressed by saying that the *contribution* of the coupling (8) is equal to N^3.

In general (for an arbitrary k, and for an arbitrary coupling), the contribution is equal to N^V, where V is the number of "free indices".

It seems that the question is more or less settled. But the most interesting part of the story only begins, because the above equalities among indices have a fascinating geometric interpretation.

4. MAPS

4.1. Graphs, Surfaces and Maps: Basic Notions

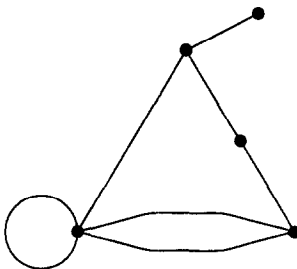Basic facts concerning the combinatorics of maps may be found, for example, in [13].

Figure 2. A graph.

4.1.1. Graphs

We accept graphs with loops and multiple edges, as well as graphs with vertices of degree 1 and 2 (see Figure 2). But usually we will only consider connected graphs.

4.1.2. Surfaces

We will consider compact oriented 2-dimensional manifolds without boundary. From now on we will simply use the word *surface* to denote such an object. The *genus* of a surface is the number $g \geq 0$ of "holes" in it, or the number of "handles"; see Figure 3.

Figure 3. Surfaces of genus $g = 0, 1, 2$.

The surfaces are classified according to their genera: every surface has genus, which is a nonnegative integer, and there is only one, up to a homeomorphism surface of any given genus.

4.1.3. Maps

A map is a graph *drawn* on a surface. The difference between the notion of a graph and that of a map is rather subtle and does not strike one's eye. The reason is that whenever we deal with a graph, we usually draw it. But the mere fact of drawing a graph provides it with some additional mathematical structure.

Let us be more precise.

DEFINITION 4.1. A *map* is a graph which is "drawn" on (imbedded into) a surface in such a way that:

- the edges do not intersect;
- if we "cut" the surface along the edges, we get a disjoint union of sets which are homeomorphic to an open disk (these sets are called *faces* of the map).

The second condition implies the connectedness of the graph. It is important to notice that the graph itself does not have any faces, but only vertices and edges. The *degree of a vertex* is the number of edges incident to it (a loop incident to a vertex is counted twice). The *degree of a face* is the number of its boundary edges (an edge may be adjacent to a face "from both sides"; then it is counted twice).

PROPOSITION 4.2. *The following quantity, which is called the Euler characteristic of a map,*

$$\chi = \#(vertices) - \#(edges) + \#(faces) = 2 - 2g,$$

does not depend on the map itself but only on the genus of the surface on which it is drawn.

REMARK 4.3. Sometimes it is necessary to consider nonconnected maps. In such a case, the maps are considered as drawn not on a single surface, but on several disjoint surfaces (one surface for each component). The Euler characteristic is additive, that is, the Euler characteristic of the whole map is the sum of the Euler characteristics of its components (because the numbers of vertices, edges, and faces are additive), while the genus is not additive.

4.2. Examples

EXAMPLE 4.4. In Figure 4, we see the same graph, but two different maps (we consider these pictures as drawn on the sphere, and not on the plane). The left-hand map has two faces, of degree 5 (the outer face) and 1, while the right-hand map has both faces of degree 3.
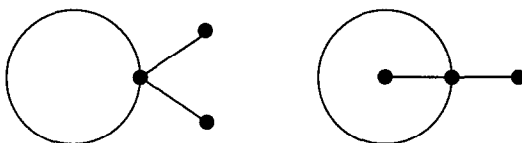

Figure 4. One graph, but two maps.

EXAMPLE 4.5. In Figure 5, we see the same graph, namely, the complete graph K_4 (it has 4 vertices, and each vertex is connected to each other). The picture shows how this graph gives rise to two maps of different genera. The left-hand picture may be considered as a map on the sphere; it has 4 faces, all of them being triangles. On the right-hand picture the same graph K_4 is drawn on the torus. This map has 2 faces, one of degree 4, the other one of degree 8.
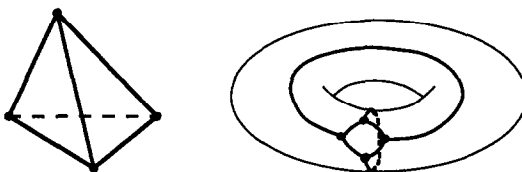

Figure 5. The same graph produces maps of different genus.

EXAMPLE 4.6. The picture shown in Figure 6 is not a map at all: one of its "faces" is not homeomorphic to the disk but possesses a handle.
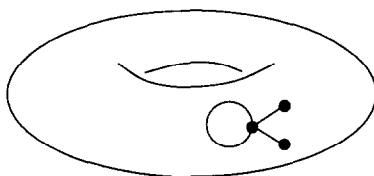

Figure 6. This is not a map.

4.3. Combinatorial Description of Maps

What kind of additional information do we need to add to a graph in order to get a map?

Consider a small neighborhood of a vertex of a graph. It consists of "ends" of edges (each edge has two ends; if an edge is a loop, both of its ends belong to the vicinity of the vertex). For a graph, this "set of edge ends" is indeed a *set*, without any inner structure in it. In order to describe a map, we must introduce a *cyclic order* on this set.

Indeed, if a graph is already drawn on a surface, the cyclic order of edges around each vertex is determined by the (counter-clockwise) orientation of the surface. But what is even more important, this information is also sufficient to reconstruct the map.

PROPOSITION 4.7. *Any given cyclic order of the ends of edges of a graph around each vertex (chosen arbitrarily and independently at each vertex) uniquely determines the imbedding of the graph into a surface, i.e., a map.*

This result is usually attributed to Edmonds [14], though according to certain authors it goes back to Heffter [15] or even to Hamilton [16]. Anyway, it is now widely known to combinatorialists working with maps. Instead of giving a formal proof of the proposition, we give a simple geometric construction which explains how the map is reconstructed from the cyclic orders of edges around vertices.
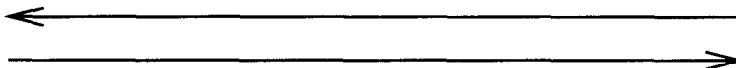
Let us represent an edge as a "two-way" street.

Figure 7. A two-way street.

Let us represent a vertex as a crossroad made as a "roundabout" (Figure 8). This geometric image of a roundabout corresponds nicely to the formal notion of cyclic order of edges around a vertex. Now, let us move through this city, always respecting the following rule: *having come to a crossroad, leave it by the first street to your right.*
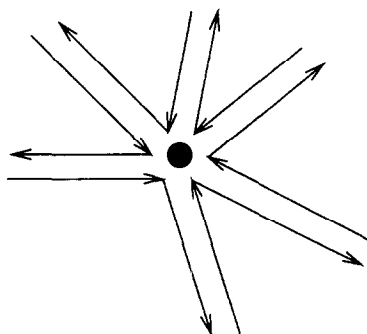

Figure 8. A "roundabout" crossroad.

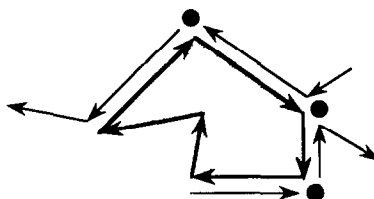Then all possible routes form a set of disjoint cycles.

Figure 9. A face.

We may consider these cycles as faces of a map. We may just *call* them faces, in a purely formal manner, filling them by open polygonal sets. Note, that the streets at each crossroad are oriented counter-clockwise, but the boundaries of faces are oriented clockwise.

This construction is often called a "ribbon graph", or a "fat-graph".

REMARK 4.8. For the graph shown in Figure 6, the cyclic order of the edges around vertices is also prescribed by the imbedding. But when we reconstruct the map structure from this cyclic order, we must glue a simple pentagon as its outer face. The map thus becomes of genus zero: the handle has appeared illegally.

In a dual approach, we may start from a number of faces, that is, polygons with their boundaries oriented clockwise, with the total number of edges being even, and then glue the edges pair-wise in an arbitrary (but connected) way, always respecting the rule that the arrows glued together must point in opposite directions, as in Figure 10. Then the cyclic order of edges around a vertex is determined by the rule: the next edge is the one that we may access to by the interior of the face.
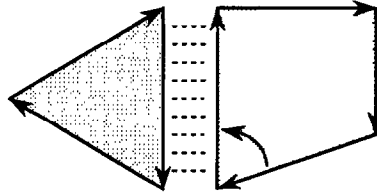
Figure 10. Gluing of faces.

The number of cyclic orders of edges of a graph at a vertex v of degree d_v is $(d_v - 1)!$. Therefore, the total number of edge orderings of the graph is equal to $\prod(d_v - 1)!$, where the product is taken over the set of vertices, and every such ordering leads to a map. There exist many unusual imbeddings even for well-known graphs. The reader may entertain him or herself by trying to imbed, for example, the graph of the cube into the torus (in a regular way, so that all the faces be hexagons), or the graph of the icosahedron into the surface of genus 4 (also regularly, so that all the faces be pentagons).

5. ONE FACE MAPS

5.1. An Enumeration Problem

Consider a polygon with $2k$ sides. This polygon will serve as the only face of the maps we are going to construct. Orient the polygon perimeter clockwise, and label its vertices by labels $i_1, i_2, \ldots, i_{2k}$. Now, let us split the sides of the polygon into pairs in an arbitrary way, and glue together the sides belonging to each pair, always respecting the rule "end to the head". What we get is a map.

There are $(2k - 1)!!$ ways of splitting the set of the *labeled* edges into pairs. (Indeed, couple the first edge with any of the other $2k - 1$ ones; than couple the first "nonappointed" edge to any one of the $2k - 3$ edges that left, etc.). Hence there are $(2k - 1)!!$ corresponding maps. Our goal is to enumerate them according to their genus. It is easy to see that the maximal genus is equal to $[k/2]$, where the square brackets denote the integer part. So we are looking for the numbers

$$\varepsilon_g(k) := \#(\text{gluings of } 2k\text{-gon of genus } g), \qquad \sum_{g=0}^{[k/2]} \varepsilon_g(k) = (2k - 1)!!.$$

This problem has all the attributes of a worthy problem: it is beautiful, difficult, and important. The answer for the genus zero has been well known for a long time: it is given by Catalan numbers,

$$\varepsilon_0(k) = C_k = \frac{1}{k+1}\binom{2k}{k}. \tag{9}$$

Some preliminary results for arbitrary genus were obtained in [17]. The problem was completely solved in [18].

Let us see how to determine the genus of a map in question. In order to compute its Euler characteristic, we must know the number of vertices V, the number of edges E, and the number of faces F. The number of faces is the easiest to found: our map has only one face *by construction*,

namely, the polygon itself; so $F = 1$. As to the number of edges, it is equal to $E = k$, because the original polygon had $2k$ sides, and they were glued together in pairs. The only unknown parameter is the number V of vertices. We have

$$\chi = V - E + F = V - k + 1 = 2 - 2g,$$

hence,

$$V = k + 1 - 2g.$$

Let us look how to determine V in a particular example.

EXAMPLE 5.1. Consider a polygon with 8 sides, and glue its sides in pairs as is shown in Figure 11.
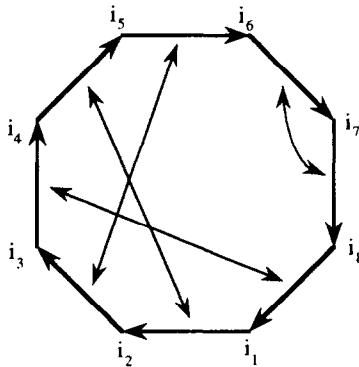

Figure 11. A polygon whose sides are glued together in pairs.

We remind to the reader that the arrows must be glued together in the opposite directions. The fact that the side labeled $i_1 i_2$ is glued to the one labeled $i_4 i_5$, means that the polygon vertex i_1 is identified with i_5, as well as i_2 is identified with i_4. We express this by writing

$$i_1 = i_5, \qquad i_2 = i_4.$$

In the same way, the side $i_2 i_3$ is glued to $i_5 i_6$, hence,

$$i_2 = i_6, \qquad i_3 = i_5,$$

and the two remaining identifications give us

$$i_3 = i_1, \qquad i_4 = i_8, \qquad i_6 = i_8, \qquad i_7 = i_7.$$

Finally, we see that the 8 original vertices of the polygon are identified in the following way (thus, producing 3 vertices of the corresponding map):

$$i_1 = i_5 = i_3 = i_1,$$
$$i_2 = i_4 = i_8 = i_6 = i_2,$$
$$i_7 = i_7.$$

We hope that by now the reader has recognized the equalities of Example 3.3. This leads to a geometric interpretation of the Wick coupling described in Section 3.2.

5.2. Back to Gaussian Integrals

We will "graphically represent" the integral of $\text{tr}(H^{2k})$ by a polygon with $2k$ sides whose vertices are labeled by $i_1, i_2, \ldots, i_{2k}$ clockwise. The cyclic structure of indices in (7) is reflected in the cyclic arrangement of labels around the polygon. Each of $(2k-1)!!$ Wick couplings is represented by one of the $(2k-1)!!$ gluings of the sides of the polygon, that is, by the corresponding one-face map. (In the language of physics, one would say that this map serves as a *Feynman diagram* for our integral.) The number V of free indices of a Wick coupling is equal to the number $V = k + 1 - 2g$ of vertices of the corresponding map. The contribution of a coupling (map, diagram, whatever) into the integral is equal to N^V.

We have established the following theorem [18].

THEOREM 5.2. *Let* $f(k, N)$ *be*

$$f(k, N) := \left\langle \text{tr}\left(H^{2k}\right) \right\rangle = \int_{\mathcal{H}_N} \text{tr}\left(H^{2k}\right) \, d\mu(H),$$

where the integral is taken over the space $\mathcal{H}_N$ *of Hermitian* $(N \times N)$*-matrices with respect to the Gaussian measure (6). Then*

$$f(k, N) = \sum_{g=0}^{[k/2]} \varepsilon_g(k) N^{k+1-2g} = N^{k+1} \sum_{g=0}^{[k/2]} \varepsilon_g(k) \left(\frac{1}{N^2}\right)^g,$$

where $\varepsilon_g(k)$ *is the number of labeled one-face maps of genus* g *with* k *edges.*

REMARK 5.3. The function $f(k, N)$ may be considered as a kind of a generating function for the numbers $\varepsilon_g(k)$, where the role of a formal parameter is played by the quantity $1/N^2$.

5.3. Results and Discussion

We have explained only a very small part of the paper [18], where the method of matrix integrals was independently rediscovered. Now, one must compute the integral, which is not at all an easy task:

$$f(k, N) = (2k - 1)!! \sum_{m \geq 1} \binom{N}{m} \binom{k}{m-1} 2^{m-1}.$$

The next step is to extract from the above formula an information concerning the numbers $\varepsilon_g(k)$. The final result takes the form of a recurrent relation

$$\varepsilon_g(k) = \frac{4k-2}{k+1} \varepsilon_g(k-1) + \frac{(k-1)(2k-1)(2k-3)}{k+1} \varepsilon_{g-1}(k-2), \tag{10}$$

with the boundary conditions

$$\varepsilon_g(0) = \begin{cases} 1, & \text{if } g = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The table of values of $\varepsilon_g(k)$ may be found in [18].

Relations (10) are so simple that one is tempted to look for a purely combinatorial proof, without matrix integrals. The first such proof, using characters of the symmetric group, was given in [19]. It was still very complicated. A considerable simplification is achieved in [20]. But a convincing purely geometrical proof still does not exist.

The enumerative result for $\varepsilon_g(k)$ in [18] has only served as a combinatorial lemma in a study of a much more complicated object, namely, the moduli space of algebraic curves. The method of computation of its (virtual) Euler characteristic based on this lemma was later considerably simplified by Kontsevich [21] who reduced the problem to the computation of a one-dimensional Gaussian integral. In cartographic terms, the main result of [21] may be formulated as follows.

For any map m let $V(m)$ and $E(m)$ denote the number of its vertices and edges, respectively. Let $\mathcal{M}_n$ be the set of maps such that *the degree of each vertex is at least* 3, and $E(m) - V(m) = n$ (we no longer demand the number of faces to be equal to one). Note, that this set $\mathcal{M}_n$ is finite.

THEOREM 5.4. *The following formula is valid:*

$$\sum_{m \in \mathcal{M}_n} \frac{(-1)^{V(m)}}{|\mathrm{Aut}(m)|} = \frac{B_{n+1}}{n(n+1)},$$

where $|\mathrm{Aut}(m)|$ *is an order of the automorphism group of the map* m, *and* B_{n+1} *is a Bernoulli number.*

6. ONE MORE INTEGRAL

Let us consider the following more complicated integral:

$$Z(t, N) := \left\langle \exp\left\{ -\frac{t}{N} \mathrm{tr}(H^4) \right\} \right\rangle, \tag{11}$$

(historically, this integral was studied earlier than the previous one, see [4,5]). On the one hand, $\mathrm{tr}(H^4)$ is simpler than $\mathrm{tr}(H^{2k})$; on the other hand, we have here an exponential function, while we only know how to integrate polynomials.

REMARK 6.1. Integral (11) is not trivial even in the one-dimensional case:

$$z(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2/2) - tx^4} \, dx.$$

We see that the latter integral converges for all $t \geq 0$ and diverges for all $t < 0$. Hence, it cannot be analytic at $t = 0$. In physics, this reasoning is called "Dyson's argument". The integral may be expressed in terms of Bessel functions.

6.1. Perturbative Series

If we must deal with the exponential function but are only able to integrate polynomials, then it is clear what to do: we must first expand the exponential function into a power series:

$$\exp\left\{ -\frac{t}{N} \mathrm{tr}\left(H^4\right) \right\} = \sum_{n \geq 0} \frac{1}{n!} \left(-\frac{t}{N} \right)^n \left(\mathrm{tr}\left(H^4\right) \right)^n.$$

And now we must struggle only against $(\mathrm{tr}(H^4))^n$. We have

$$\mathrm{tr}\left(H^4\right) = \sum_{i,j,k,l=1}^{N} h_{ij} h_{jk} h_{kl} h_{li},$$

hence,

$$\left(\mathrm{tr}\left(H^4\right)\right)^n = \sum h_{i_1 j_1} h_{j_1 k_1} h_{k_1 l_1} h_{l_1 i_1} h_{i_2 j_2} h_{j_2 k_2} \ldots h_{k_n l_n} h_{l_n i_n},$$

where the sum is taken over all the N^{4n} possible combinations of indices. And now we must apply the Wick formula to this incredibly cumbersome expression! Feynman diagrams are indeed an invaluable tool for overcoming many-storeyed indices, i.e., to present them in a geometric way.
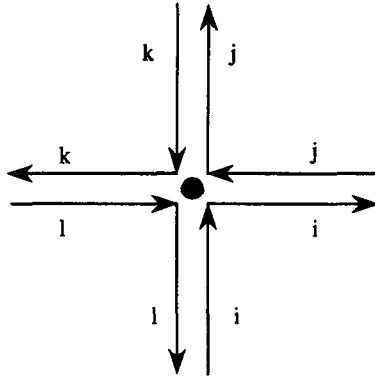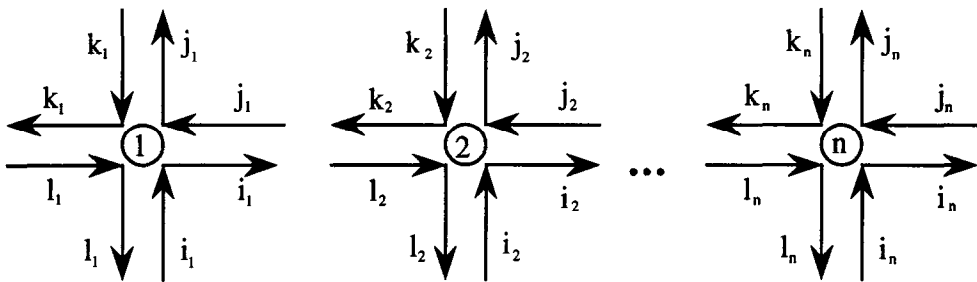
Figure 12. An image for $\text{tr}(H^4)$.

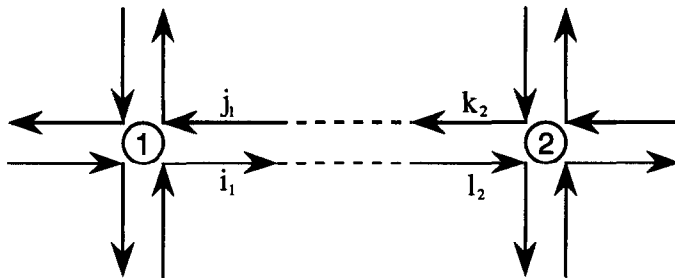Figure 13. An image for $(\text{tr}(H^4))^n$.

Figure 14. Wick coupling.

6.2. Feynman Diagrams

Let us represent the $\text{tr}(H^4)$ by the following geometric image (a "crossroad"), Figure 12.

The power $(\text{tr}(H^4))^n$ is represented by n crossroads as in Figure 13.

How does the Wick coupling look? Suppose that $h_{i_1 j_1}$ is coupled with $h_{k_2 l_2}$. We draw this fact in the following manner shown in Figure 14.

The equalities

$$i_1 = l_2, \qquad j_1 = k_2,$$

which assure that the result of the coupling is equal to 1 (and not to 0) are translated geometrically by asserting that i_1 is "the same route" as l_2, and j_1 is the same route as k_2.

When the procedure of coupling is finished, what we get is the following object.

(1) It is a graph with $V = n$ vertices, each vertex being of degree 4.

(2) The number of edges is $E = 4n/2 = 2n$; the edges have a prescribed cyclic order around each vertex.

(3) The equalities between indices $i_1 = l_2 = \cdots = i_1$, $j_1 = k_2 = \cdots j_1$, etc., constitute "chains" or "cycles", which correspond to the *faces*! Hence, we have the final point.

(4) The contribution of a diagram into the integral (11) is equal to N^F, where F is the number of faces of the diagram.

Having in mind that the Euler characteristic is

$$\chi = V - E + F = n - 2n + F = -n + F \Rightarrow F = n + \chi,$$

we get the following result:

$$Z(t, N) = \sum_{n \geq 0} \frac{1}{n!} \left(-\frac{t}{N} \right)^n \sum N^{n+\chi} = \sum_{n \geq 0} \frac{1}{n!} (-t)^n \sum N^\chi,$$

where the internal sum is taken over all the diagrams (couplings).

But this is not yet the result for which we are looking.

6.3. Logarithm

We have missed a very important point: our diagrams do not have to be connected. Quite on the contrary: we must consider all the possible matchings, disconnected ones included. (We tried to avoid the term "map" for this object on purpose.) In order to find a more interesting combinatorial result, we must yet perform four more operations.

(1) Take the logarithm of $Z(t, N)$. This operation is very well known in combinatorics; see, for example, [22]. If one has an *exponential* generating function for a class of *labeled* objects, then its logarithm is the generating function of the *connected* objects of the same type.

The word "exponential" means "with $n!$ in the denominator". The term "labeled" should be explained at more length. But we will limit ourselves to a single remark. One may consider the objects (such as graphs) with additional structures (such as cyclic orders of edges around vertices) and with some properties imposed (such as the prescribed vertex degrees). But one must be careful: a property has a legal right to be imposed when *it is valid for an object if and only if it is valid for all its connected components*. For example, the property:

every vertex of a graph is colored in one of k colors

is acceptable; on the contrary, the property:

every vertex of a graph is colored in one of k colors, and all k colors are used

is not good, because it may be valid for a graph as a whole but not valid for some of its components. See the discussion of this subtle point in [23].

Concerning the proof, we may once more use the formula for the logarithm of a power series of Proposition 2.11, with a slight change of the interpretation. First, we must specialize the formula from multi-indices to simple (one-dimensional) indices. Second, we must generalize the notion of "coefficients" (s_α or m_α), considering them as "weights" taking values in a ring. The weight of an object must be the product of the weights of its connected components.

In our case, the weight of a diagram is equal to N^χ, and the multiplicativity of this weight is implied by the additivity of the Euler characteristic χ, see Remark 4.3. Thus, finally,

$$\log Z(t, N) = \sum_{n \geq 1} \frac{1}{n!} (-t)^n \sum N^{2-2g},$$

where the second sum this time is taken over all the *connected* diagrams, and g is the genus of a diagram.

(2) Let us now divide the result by N^2, in order to have weights of the form N^{-2g}.

(3) The third operation is to change the sign preceding the whole thing, that is, to take the "minus logarithm". This operation is not very meaningful mathematically, but it is meaningful from the physical point of view. As one physicist has put it, "Partition function does not have its own physical meaning, but its logarithm has the meaning of energy". Function $Z(t, N)$ obviously has the meaning of a partition function (see discussion below), i.e., it is a "sum" (rather an integral) of the exponents of "minus energy". This is yet another facet of the logarithm of a power series.

(4) The last operation is the changing of the summation order, to collect the diagrams of the same genus. Finally, we get the following theorem (see [4,5]).

THEOREM 6.2. *Let* $E(t, N)$ *be*

$$E(t, N) := -\frac{1}{N^2} \log Z(t, N) = -\frac{1}{N^2} \log \left\langle \exp\left\{ -\frac{t}{N} \operatorname{tr}\left(H^4\right) \right\} \right\rangle.$$

Then,

$$E(t, N) = \sum_{g \geq 0} \left(\frac{1}{N^2}\right)^g E_g(t),$$

where

$$E_g(t) = -\sum_{n \geq 1} \frac{1}{n!} (-t)^n K_g(n)$$

and $K_g(n)$ *is the number of connected diagrams of genus* g *with* n *vertices.*

Once again $1/N^2$ plays the role of a formal parameter in the genus expansion.

REMARK 6.3. Let us explain more carefully the notion of a diagram used in the above theorem. A diagram is:

• a connected graph with n vertices, all the vertices being of degree 4,
• vertices are labeled by numbers $1, 2, \ldots, n$,
• the edges incident to each vertex are labeled by numbers 1,2,3,4 and this labeling corresponds to their cyclic order.

The following picture gives, for each graph, the total number of its possible labelings (that is, corresponding diagrams), and, in parentheses, the number of planar ones. So, for example, $K_0(1) = 2$, $K_0(2) = 4 + 32 = 36$, $K_0(3) = 64 + 768 + 384 + 512 = 1728$.
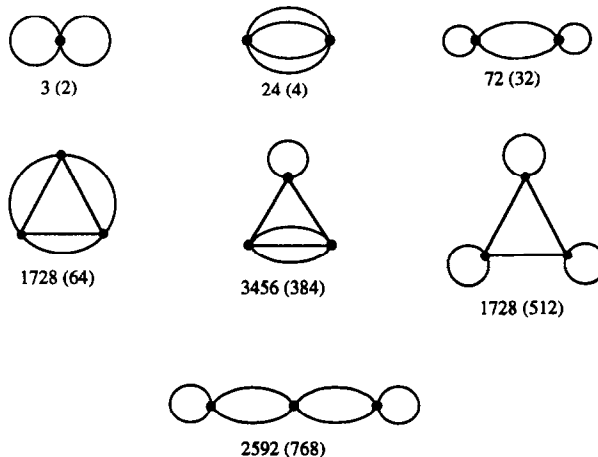

Figure 15. The number of diagrams (respectively, planar diagrams) corresponding to a graph.

REMARK 6.4. The function $E(t, N)$ is not analytic at $t = 0$ (sf. Remark 6.1), but the functions $E_g(t)$ are, with the radius of convergence tending to zero as $g \to \infty$. Therefore, a more accurate formulation of Theorem 6.2 would be as follows.

For any $k \geq 0$, there exists such a $t_0 > 0$ that, as $N \to \infty$, the function $E(t, N)$ admits an asymptotic expansion

$$E(t, N) = \sum_{g=0}^{k} \left(\frac{1}{N^2}\right)^g E_g(t) + o\left(N^{-2k}\right),$$

the functions $E_g(t)$, $g = 0, \ldots, k$ being analytic in the circle $|t| < t_0$.

6.4. Results and Discussion

6.4.1. Some enumerative formulae

The following enumerative results were established in several papers, and then brought together in [8]. Denote

$$u := \frac{-1 + \sqrt{1 + 48t}}{24t} = \sum_{n \geq 0} (-1)^n (12t)^n C_n,$$

an "almost" generating function for Catalan numbers C_n, see (9).

GENUS $g = 0$.

$$E_0(t) = \frac{1}{24}(u - 1)(9 - u) - \frac{1}{2} \log u.$$

The coefficients $K_0(n)$ of this generating function are given by the formula

$$K_0(n) = \frac{(2n - 1)! \, 12^n}{(n + 2)!},$$

the first values being 2, 36, 1728, 145156, etc. (cf. Remark 6.3).

GENUS $g = 1$.

$$E_1(t) = \frac{1}{12} \log(2 - u)$$

and

$$K_1(n) = \frac{((2^n \cdot n!)^2 - (2n)!) 12^n}{24n \cdot n!},$$

the first values being 1, 60, 6336, etc.

GENUS $g = 2$.

$$E_2(t) = \frac{1}{6!} \frac{(1 - u)^3}{(2 - u)^5} \left(82 + 21u - 3u^2\right).$$

GENUS $g \geq 2$. For an arbitrary genus $g \geq 2$

$$E_g(t) = \frac{(1 - u)^{2g-1}}{(2 - u)^{5(g-1)}} P_g(u),$$

where $P_g(u)$ is a polynomial.

6.4.2. Why is it physics?

The reader may wonder why all this is physics? What is the physical meaning of the above results?

The author is not in a position to give here a detailed account on that, due to his profound ignorance in physics. I am, however, able to reproduce some words I have heard during several physical talks I have attended. There are two physical images that come to mind. They are, obviously, mathematically related to each other (as this paper shows). Whether they are also physically connected, it is not for me to judge.

String Theory

In classical mechanics, a point particle that moves from A to B chooses a trajectory $x(t)$ that minimizes the functional of action $S[x]$. In quantum mechanics (more exactly, in Feynman path integrals approach), a particle moves from A to B along all possible trajectories, the contribution of each trajectory into the "amplitude of probability" being $e^{-(i/\hbar)S[x]}$. In order to compute the amplitude (and other physical quantities), one must evaluate an integral over the space of all possible trajectories. One of the techniques of such a computation consists in replacing trajectories by broken lines, thus reducing the problem to a finite-dimensional integration. If in the process of its evolution in time the particle breaks down into two, and/or two particles collide and form one particle, the space of trajectories becomes more complicated (see Figure 16).
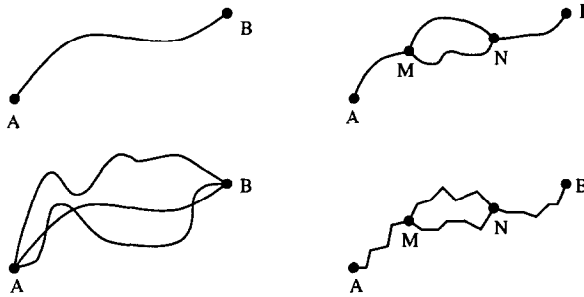

Figure 16. Feynman paths and their discrete approximations.

In the string theory approach, a particle is no more point-like, but is a small circle ("string"). As this circle evolves in time, a two-dimensional surface appears, which serves as an analog of a trajectory. The process of breaking down/colliding is described by the addition of a new handle to the surface (Figure 17).
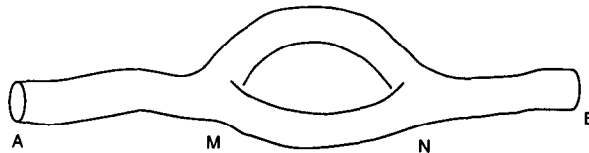

Figure 17. Evolution of a string in time.

The path integrals are now replaced by integrals over the space of two-dimensional surfaces. There are two main strategies of reducing these integrals to finite dimensional ones (miraculously, both strategies lead to the same physical results). One consists in considering a surface as a complex algebraic curve. It is well known that the space of complex algebraic curves (or, more accurately, the space of parameters describing them, the so-called moduli space of curves) is finite-dimensional, its complex dimension being $3g - 3$ for $g \geq 2$, 1 for $g = 1$, and 0 for $g = 0$. Thus, instead of integrating over the space of surfaces, one may integrate over the moduli space.

Another approach consists in replacing a surface by its discrete approximation, that is, by a map! Thus, enumeration of maps becomes relevant to this physical model, being an approximate computation of Feynman path integrals over the space of "two-dimensional paths". Many physical papers cited in our bibliography compute some parameters of physical interest, such as, for example, "string susceptibility".

Quantum Field Models

A general scheme according to which the models of quantum fields theory are constructed is as follows. There is a space X that is a model of the Universe. Fields are functions $f : X \to Y$

taking values in a space Y. There is a measure on the space $\mathcal{F}$ of all fields. The general form of the measure is

$$\exp\{-(\text{quadratic functional of } f) - (\text{interaction term})\},$$

where interaction term is nonquadratic. When there is no interaction, the measure becomes Gaussian (as it becomes exponent of a quadratic term), and it is called "free field". The interaction term may also depend on some parameters. Then we must integrate with respect to this measure, and study the dependence of the results on parameters.

Obviously, the matrix model considered above has all the ingredients of this general scheme. The role of the space Y is played by $\mathcal{H}_N$, the space of Hermitian matrices. The free field and the corresponding Gaussian measure are described by the quadratic functional $(1/2)\,\text{tr}(H^2)$. The interaction term is $(t/N)\,\text{tr}(H^4)$. The most unusual part of the story is, however, the space X: in our model the Universe consists of one point!

To my mind, this feature of the model shows in a very striking way the level of mathematical nontriviality of modern theoretical physics. Even the simplest imaginable models may lead to very complicated mathematical considerations.

7. FURTHER RESULTS, QUESTIONS, AND DISCUSSION

7.1. A Variety of Integrals

There are many other matrix integrals, computed or otherwise studied in a vast literature. It would be very difficult to explain all of them in full detail, but it is worth making a brief survey.

(1) Replacing H^4 by H^3 gives rise to the enumeration of triangulations (or, dually, of trivalent graphs); see, for example, [24].

(2) More complicated "trace polynomials" of the form $\prod_i(\text{tr}(H^{k_i}))^{m_i}$ are integrated in [25]. These enumerate maps with prescribed face degrees.

(3) Considering an "interaction potential" of the form

$$U(H) := H + \frac{1}{2}H^2 + \frac{1}{3}H^3 + \cdots = -\log(1 - H),$$

is used to enumerate maps with arbitrary vertex degrees, see [26].

(4) Considering a Gaussian measure with nonzero mean may help to enumerate maps without loops (Kazakov, private communication).

(5) Integrals over the space of real symmetric matrices are related to maps on nonoriented surfaces (see forthcoming works of Goulden).

(6) The "two-matrix" model is very interesting physically (but much more complicated computationally), see [5]. The Gaussian measure in $\mathcal{H}_N \times \mathcal{H}_N$ is introduced by means of the quadratic form

$$\text{tr}\left(H_1^2 + H_2^2 - 2cH_1H_2\right).$$

While considering Feynman diagrams for this measure, we must assign to each vertex either H_1 or H_2, the contribution of the diagram depending on this assignment. Thus, the diagrams are configurations of the Ising model on the graph in question.

While the two-matrix model is solved, its direct generalization, the n-matrix model, is not.

(7) A kind of a Potts model suggested by Kazanov is related in [27] to the problem of enumerating "meanders". This model involves $2q$ matrices, q being one of the parameters of the model. For later development, see [28].

(8) In [29], the first attack is launched against the most intriguing (but undoubtedly, very difficult) problem of enumerating maps according to both the set of vertex degrees and the set of face degrees.

(9) The most famous recent application of the method of matrix integrals is [30]. Kontsevich has proved a conjecture of Witten, and has shown, among other things, that some generating functions for maps satisfy higher analogs of the Korteweg-de Vries equation.

(10) Some additional references: for various methods of computation of matrix integrals, see [31–33]; for experimental numerical results, see [34]; for various recent developments, see [7,35–37].

7.2. First Glimpses of the Computation of Integrals

The fundamental observation is that the density (6), as well as all the functions of H we would like to integrate, are unitary invariant. This means that we may introduce the so-called "polar coordinates" in the space $\mathcal{H}_N$, representing each matrix $H \in \mathcal{H}_N$ by a pair (U, Λ), where $U \in \mathcal{U}(N)$ is a unitary matrix, Λ is a diagonal matrix, and

$$H = U \Lambda U^{-1}. \tag{12}$$

Then the density and all the integrands depend only on Λ and do not depend on U.

These "coordinates" are not coordinates properly speaking, as the representation (12) is ambiguous. The matrix Λ is determined to within permutation of its diagonal entries λ_i, $i = 1, \ldots, N$ (and the corresponding permutation of the columns of U), and the matrix U, in the case of all λ_i different (the opposite case has measure 0), to within multiplication by a diagonal matrix of the form

$$\begin{pmatrix} e^{i\varphi_1} & & & \\ & e^{i\varphi_2} & & \\ & & \ddots & \\ & & & e^{i\varphi_N} \end{pmatrix}.$$

This ambiguity is responsible for the factor $1/((2\pi)^N \cdot N!)$ in the formula below.

The following proposition asserts two major facts: first, that we may "integrate out" the unitary part, and second, that the Jacobian of this change of variables is equal to the square of the Vandermonde determinant.

PROPOSITION 7.1. *Let a function $f(H)$ be unitary invariant. Then,*

$$\int_{\mathcal{H}_N} f(H) \, d\mu(H) = \frac{1}{(2\pi)^N \cdot N!} \mathrm{vol}(\mathcal{U}(N)) \int_{\mathbb{R}^N} f(\Lambda) \prod_{i<j} (\lambda_i - \lambda_j)^2 \, d\nu(\Lambda),$$

where $\mathrm{vol}(\mathcal{U}(N))$ *is the volume of the unitary group, and*

$$d\nu(\Lambda) := (2\pi)^{-N/2} \exp \left\{ -\frac{1}{2} \sum_{i=1}^N \lambda_i^2 \right\} \prod_{i=1}^N d\lambda_i$$

is the standard Gaussian measure in $\mathbb{R}^N$.

The volume of the unitary group is induced by the metrics of its imbedding as a N^2-dimensional surface into the space of complex $(N \times N)$-matrices, and this latter space is considered as $\mathbb{R}^{2N^2}$.

This volume was calculated many times, for different purposes. It is equal to

$$\mathrm{vol}(\mathcal{U}(N)) = \frac{(2\pi)^{(N^2+N)/2}}{\prod_{k=1}^{N-1} k!}.$$

The functions to integrate are: $f(\Lambda) = \mathrm{tr}(\Lambda^{2k}) = \sum \lambda_i^{2k}$ for one of the models considered above; $f(\Lambda) = \exp\{-(t/N)\mathrm{tr}(\Lambda^4)\} = \exp(-(t/N)\sum \lambda_i^4)$ for the other. But we would propose, to begin with, to compute the integral of the function $f(\Lambda) \equiv 1$. The result is known in advance, of course:

it is equal to 1 (the measure is normalized!). Being compared to the formula for the volume of $\mathcal{U}(N)$, this result is equivalent to the following identity:

$$\int_{\mathbb{R}^N} \prod_{i<j} (\lambda_i - \lambda_j)^2 \, d\nu(\Lambda) = \prod_{k=1}^{N} k!.$$

The right-hand side suggests some (yet unknown) combinatorial meaning. The left-hand side contains an integral with respect to the standard Gaussian measure in $\mathbb{R}^N$, of a polynomial that seems to be specially prepared for an application of the Wick formula, as it is already split into linear factors. However, the only way known to the author to prove this identity is not combinatorial: it involves Hermite polynomials.

7.3. Some Questions, Not Necessarily Open

(1) Witten [38] characterizes the works [39–41] as a "spectacular success". All these papers deal with an asymptotic enumeration of maps of arbitrary genus. The results obtained seem to be very similar to those obtained in [42], which also treats maps of arbitrary genus (this last paper is cited in [40]). The question is, what are the physical implications of the works of Bender and his co-authors on asymptotic map enumeration?

(2) A very general nature of Proposition 2.11 suggests paying a closer attention at a general theory of integration of power series with respect to various measures. A most obvious obstacle is the theorem of Marcinkiewicz (see, for example, [43]) stating that if the logarithm of the characteristic function of a probabilistic distribution is a polynomial, then it is a quadratic polynomial, and the distribution is Gaussian. So, in a case other than Gaussian, we must deal with an infinite series. Or maybe it is worth considering a convolution not with a probabilistic measure but with an arbitrary function which is an inverse Fourier transform of a polynomial (such as the Airy function). As an example of an interesting non-Gaussian distribution in the space of matrices we may mention the Wishart distribution, a matrix analog of the gamma-distribution on the line; see [44], where one may also find many interesting results concerning matrix integrals.

(3) The world of maps is full of various bijections; see, for example, [45]. For example, arbitrary maps with n edges are in bijection with maps with n vertices of degree 4: just put a new vertex in the middle of each edge. The question is, do these bijections lead to some interesting identities for matrix integrals?

REFERENCES

1. L. Schneps, Editor, *The Grothendieck Theory of Dessins d'Enfants*, London Math. Soc. Lecture Notes Series, Vol. 200, Cambridge Univ. Press, (1994).
2. D. Bar-Natan, On the Vassiliev knot invariants, *Topology* **34** (2), 423–472 (1995).
3. A.K. Zvonkin, How to draw a group, In *Formal Power Series and Algebraic Combinatorics 95*, (Edited by B. Leclerc and J.-Y. Thibon), pp. 601–611, Paris (May 1995).
4. E. Brézin, C. Itzykson, G. Parisi and J.-B. Zuber, Planar diagrams, *Commun. Math. Phys.* **59**, 35–51 (1978).
5. C. Itzykson and J.-B. Zuber, The planar approximation II, *J. Math. Phys.* **21** (3), 411–421 (1980).
6. G. 't Hooft, A planar diagram theory for strong interactions, *Nuclear Physics B* **72**, 461–473 (1974).
7. Ph. Di Francesco, P. Ginsparg and J. Zinn-Justin, 2D gravity and random matrices, *Phys. Rep.* **254**, 1–133 (1995).
8. D. Bessis, C. Itzykson and J.-B. Zuber, Quantum field theory techniques in graphical enumeration, *Adv. in Appl. Math.* **1** (2), 109–157 (1980).
9. M. Bauer and C. Itzykson, Triangulations, In *The Grothendieck Theory of Dessins d'Enfants*, London Math. Soc. Lecture Notes Series, Vol. 200, pp. 179–236, Cambridge Univ. Press, (1994).
10. A.N. Shiryaev, *Probability*, Second Edition, Springer-Verlag, New York, (1996).
11. B. Simon, *The $P(\varphi)_2$ Euclidean (Quantum) Field Theory*, Princeton Univ. Press, (1974).
12. V.A. Malyshev and R.A. Minlos, *Gibbs Random Fields*, Kluwer, Dordrecht, (1991).
13. A.T. White, *Graphs, Groups and Surfaces*, (2nd edition), North-Holland, Amsterdam, (1984).

14. J.R. Edmonds, A combinatorial representation for polyhedral surfaces, *Notices Amer. Math. Soc.* **7**, 646 (1960).

15. L. Heffter, Über das problem der Nachbargebiete, *Math. Ann.* **38**, 477–508 (1891).

16. W.R. Hamilton, Letter to John T. Graves "On the Icosian" (17th October 1856), In W.R. Hamilton, *Mathematical Papers*, Vol. III, Algebra (Edited by H. Halberstam and R.E. Ingram), pp. 612–625, Cambridge Univ. Press, (1967).

17. T.R.S. Walsh and A.B. Lehman, Counting rooted maps by genus I, *J. of Combinat. Theory B* **13**, 192–218 (1972).

18. J. Harer and D. Zagier, The Euler characteristic of the moduli space of curves, *Invent. Math.* **85**, 457–485 (1986).

19. D.M. Jackson, Counting cycles in permutations by group characters, with an application to a topological problem, *Trans. Amer. Math. Soc.* **299** (2), 785–801 (1987).

20. D. Zagier, On the distribution of the number of cycles of elements in symmetric groups, *Nieuw. Arch. Wiskd., Ser. IV* **13** (3), 489–495 (1995).

21. M. Kontsevich, New computation of the Euler characteristic of the moduli space of curves, (in Russian), (Unpublished note), 3 (1990).

22. F. Harary and E.M. Palmer, *Graphical Enumeration*, Academic Press, New York, (1973).

23. R.C. Read and E.M. Wright, Coloured graphs: A correction and extension, *Canad. J. Math.* **22** (3), 594–596 (1970).

24. V.A. Kazakov, Bosonic strings and string field theories in one-dimensional target space, (preprint), 49 (October 1990).

25. R.C. Penner, Perturbative series and the moduli space of Riemann surfaces, *J. Diff. Geometry* **27**, 35–53 (1988).

26. I.K. Kostov and M.L. Mehta, Random surfaces of arbitrary genus: Exact results for $D = 2$ and -2 dimensions, *Physics Letters B* **189** (1/2), 118–124 (1987).

27. S.K. Lando and A.K. Zvonkin, Meanders, *Selecta Mathematica Sovietica* **11** (2), 117–144 (1992).

28. Ph. Di Francesco, O. Golinelli and E. Guitter, Meander, folding and arch statistics, (preprint), 82 (June 1995).

29. Ph. Di Francesco and C. Itzykson, A generating function for fatgraphs, (preprint), 27 (1992).

30. M. Kontsevich, Intersection theory on the moduli space of curves and matrix Airy function, *Comm. Math. Phys.* **147** (1), 1–23 (1992).

31. D. Bessis, A new method in the combinatorics of the topological expansion, *Comm. Math. Phys.* **69**, 147–163 (1979).

32. M.L. Mehta, A method of integration over matrix variables, *Comm. Math. Physics* **79**, 327–340 (1981).

33. S. Chadha, G. Mahoux and M.L. Mehta, A method of integration over matrix variables: II, *J. Phys. A: Math. Gen.* **14**, 579–586 (1981).

34. D.V. Boulatov, V.A. Kazakov, I.K. Kostov and A.A. Migdal, Analytical and numerical study of a model of dynamically triangulated random surfaces, *Nuclear Physics B* **275**, 641–686 (1986).

35. C. Itzykson and J.-B. Zuber, Matrix integration and combinatorics of modular groups, *Commun. Math. Phys.* **134**, 197–207 (1990).

36. C. Itzykson and J.-B. Zuber, Combinatorics of the modular group II. The Kontsevich integrals, *Intern. J. of Modern Physics A* **7** (23), 5661–5705 (1992).

37. Ph. Di Francesco, C. Itzykson and J.-B. Zuber, Polynomial averages in the Kontsevich model, *Commun. Math. Phys.* **151**, 193–219 (1993).

38. E. Witten, Two dimensional gravity and intersection theory on moduli space, *Surveys in Diff. Geom.* **1**, 243–310 (1991).

39. E. Brézin and V.A. Kazakov, Exactly solvable field theories of closed strings, *Physics Letters B* **236** (2), 144–150 (1990).

40. M.R. Douglas and S.H. Shenker, Strings in less than one dimension, *Nuclear Physics B* **335**, 635–654 (1990).

41. D.J. Gross and A.A. Migdal, A nonperturbative treatment of two-dimensional quantum gravity, *Nuclear Physics B* **340**, 333–365 (1990).

42. E.A. Bender and L.B. Richmond, A survey of the asymptotic behaviour of maps, *J. of Combinat. Theory B* **40**, 297–329 (1986).

43. E. Lukacs, *Characteristic Functions*, Griffin, London, (1969).

44. M.G. Kendall and A. Stuart, *The Advanced Theory of Statistics, Vol. 3: Design and Analysis, and Time-Series*, Griffin, London, (1966).

45. R. Cori, Bijective census of rooted planar maps: A survey, In *Formal Power Series and Algebraic Combinatorics 93*, (Edited by A. Barlotti, M. Delest and R. Pinzani), pp. 131–141, Florence (1993).