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A SIMPLE METHOD FOR CALCULATING A PLANET'S MEAN ANNUAL INSOLATION BY LATITUDE

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Abstract. Common methods for calculating a planet's annual insolation by latitude have relied on computationally heavy or complex computer algorithms. In this paper, we show that mean annual insolation by latitude of a planet with obliquity angle β can be found by taking the definite integral of a function of longitude. This leads to faster computations and more accurate results. We discuss differences between our method and selected computational results for insolation found in the literature.

1. Introduction. Incoming solar radiation at the top of the atmosphere is an important quantity in many areas of earths systems modeling. This physical quantity is needed in areas ranging from low dimensional energy balance models (e.g. the Budyko energy balance model [1]) to large global circulation models, GCMs, (e.g. NASA's ModelE AR5 [2]). It is common practice to compute insolation by latitude using computer algorithms. For example, NASA's latitudinal insolation calculations for ModelE AR5 rely on three FORTRAN subroutines that 1) calculate Earth's orbital parameters (eccentricity, obliquity, and longitude of perihelion) as a function of year, 2) calculate distance to the sun and declination angle as functions of time of year and orbital parameters, and 3) calculate the time integrated zentih angle as a function of the declination angle and the time interval of the day [2].

These computer calculations are useful if you are working with a grid version of a planet (as is typical in GCM's), however, to convert this information to useable data for other modeling scenarios is not always straightforward. For example, in the Budyko-Widiasih energy balance model, one must know the insolation as a function of latitude in order to make use of the model [3]. Fitting a polynomial, trigonometric function, or spline to data points given by a computer program in order to obtain such a function is unideal because it obscures the true relationship between insolation and latitude.

In the following section we highlight a method developed by McGehee and Lehman in [4] which gives the mean annual insolation by latitude for any planet as a function of obliquity and eccentricity. In the last section we discuss how the results of this method compare to results obtained from two computer simulations, one for Earth [2] and the other for Pluto [5].

2. Integral method. In "A Paleoclimate model of Ice Albedo Feedback Forced by Variations in Earth's Orbit" McGehee and Lehman develop a method to compute the mean annual insolation by latitude using only mathematical principles (Section 5 in [4]). They found that one can express mean annual insolation, $\overline{I}$, as a function

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of eccentricity e, obliquity β, and sine of latitude y by finding the insolation at any point on the Earth's surface, integrating over the course of one orbital period, then integrating over all longitudes [4]. Their results are

$$\overline{I}(e, y, \beta) = Q(e)s(y, \beta)$$

where the distribution of insolation across the sine of the latitude is given by

$$s(y, \beta) = \frac{2}{\pi^2} \int_0^{2\pi} \sqrt{1 - \left(\sqrt{1-y^2}\sin\beta\sin\gamma - y\cos\beta\right)^2}\, d\gamma \tag{2.1}$$

(where γ is longitude) and the magnitude of insolation is given by

$$Q(e) = \left[\frac{K}{16\pi a^2}\sqrt{\frac{Mm}{M+m}}\right] \frac{1}{\sqrt{1-e^2}}. \tag{2.2}$$

where a is the semi-major axis, K is the solar output in Watts, M is the mass of the sun, and m is the mass of Earth. We see that their analysis is general enough to apply to any planet orbiting a star as long as the appropriate physical parameters are known.

It is important to note that McGehee and Lehmen account for axial precession in their initial analysis. The precession dependence for annual insolation averages out over the course of a year because of their assumption that precession is constant over this time period. Earth's precession period is around 26,000 years which means that the precession angle changes by about .00024 radians each year. This small change in precession angle is negligible for the Earth. It is important to note that other planets' axial precession may exhibit resonances with the planet's orbital period, resulting in non-negligible precession.

Also, note that the distribution function (Equation 2.1) is symmetric in y. We have

$$\begin{aligned} s(-y, \beta) &= \frac{2}{\pi^2} \int_0^{2\pi} \sqrt{1 - \left(\sqrt{1-(-y)^2}\sin\beta\sin\gamma - (-y)\cos\beta\right)^2}\, d\gamma \\ &= \frac{2}{\pi^2} \int_0^{2\pi} \sqrt{1 - \left(\sqrt{1-y^2}\sin\beta\sin\gamma + y\cos\beta\right)^2}\, d\gamma \end{aligned}$$

With the change of variable $\gamma = -\tilde{\gamma}$ we get:

$$\begin{aligned} &= -\frac{2}{\pi^2} \int_0^{-2\pi} \sqrt{1 - \left(\sqrt{1-y^2}\sin\beta\sin(-\tilde{\gamma}) + y\cos\beta\right)^2}\, d\tilde{\gamma} \\ &= \frac{2}{\pi^2} \int_{-2\pi}^{0} \sqrt{1 - \left(-\sqrt{1-y^2}\sin\beta\sin(\tilde{\gamma}) + y\cos\beta\right)^2}\, d\tilde{\gamma} \\ &= \frac{2}{\pi^2} \int_{-2\pi}^{0} \sqrt{1 - \left(-\left[\sqrt{1-y^2}\sin\beta\sin(\tilde{\gamma}) - y\cos\beta\right]\right)^2}\, d\tilde{\gamma} \end{aligned}$$

and with another change of variable $\tilde{\gamma} = \alpha - 2\pi$ and using the periodicity of sine we

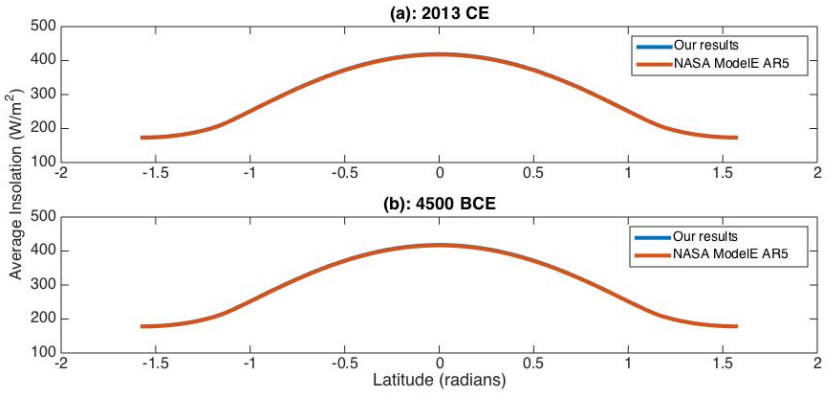

FIG. 3.1. *Insolation by latitude from the formula given in Section 2 (blue), and NASA's ModelE AR5 average insolation calculations for (orange). Plot (a) is for the year 2013 CE and plot (b) is for the year 4500 BCE. Notice that the results for both years coincide.*

get the desired result:

$$\begin{aligned}
&= \frac{2}{\pi^2}\int_0^{2\pi} \sqrt{1-\left(-\left[\sqrt{1-y^2}\sin\beta\sin(\alpha-2\pi)-y\cos\beta\right]\right)^2}\,d\alpha \\
&= \frac{2}{\pi^2}\int_0^{2\pi} \sqrt{1-\left(\sqrt{1-y^2}\sin\beta\sin(\alpha-2\pi)-y\cos\beta\right)^2}\,d\alpha \\
&= s(y,\beta)
\end{aligned}$$

This means that as long as the obliquity, precession, or eccentricity values change very little over the course of a year, the mean annual insolation will be symmetric across the y-axis (the planet's equator) no mater what the values happen to be..

3. Discrepancies between methods. In the following section we show how the method for section 2 compares with the results from two different computer programs designed to calculate insolation by latitude. We first compare it with NASA's ModelE AR5 [2] which has a program to compute insolation for the Earth and then we compare our method with a program that computes the insolation for Pluto [5].

3.1. Earth Insolation. From equations 2.2 and 2.1 above and the values of eccentricity and obliquity from the NASA fact sheet for Earth [6], we can get the annual mean insolation by latitude for 2013 CE. We plot these results in Figure 3.1 (a) along with data from the insolation calculations of ModelE AR5. From the figure we can see that our results match NASA's results for the year 2013 almost identically.

We also looked at NASA's results for the year 4500 BCE. We chose this year because Earth's period of precession is about 26,000 years and 4500 BCE is a quarter of this period. We used Laskar's calculations for obliquity and eccentricity, as these parameters change slowly over time and the values for 6,000 years ago are different from the current values. We take eccentricity to be $e = .0188425$ and obliquity to be $\beta = .421222$ [9]. Note that β is given in radians. From Figure 3.1 (b) we see that our results coincide with NASA's ModelE results again.

3.2. Pluto Insolation. From equations 2.2 and 2.1 above and the values of eccentricity and obliquity from the NASA fact sheet for Pluto [7], we can get the

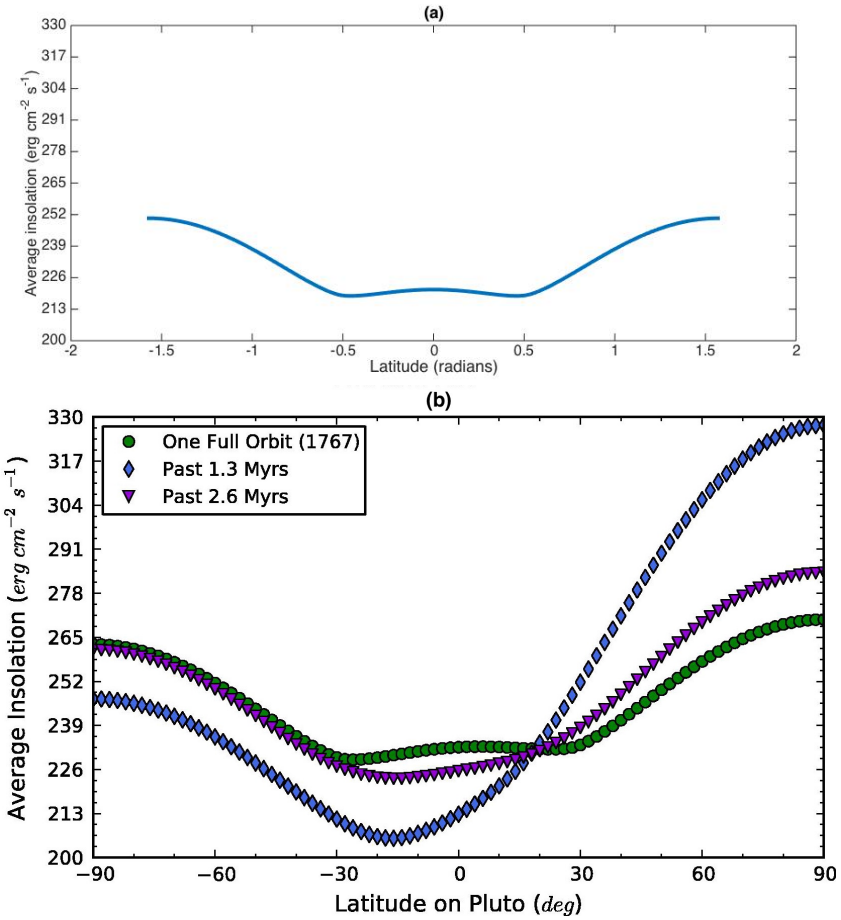

FIG. 3.2. *Insolation by latitude from the formula given in Section 2 (Figure a) and from Figure 3 in [5] (Figure b: green circles).*

annual mean insolation by latitude. We plot these results in Figure 3.2 (a) along with the results of [5] (Figure 3.2 (b)).

From the figure we can see that our results differ both quantitatively and qualitatively and. Note that we are referencing only the part of Earle and Bindles data that corresponds to the annual average by latitude, which is the green circle plot in Figure 3.2 (b). Quantitatively, the two plots differ by a constant of $13\,erg\,cm^{-2}\,s^{-1}$, but are otherwise share the same order of magnitude. On the qualitative side, we see that Earle and Binzel's results are slightly asymmetric, indicating that in Pluto's most recent year (since 1767) its north pole has received more insolation than its south pole. They claim that this slight asymmetry is due to the fact Pluto's longitude of perihelion is small (about $-3°$), resulting in Pluto's line of equinox's almost aligning with its perihelion.

Recall that our formula for insolation doesn't depend on the precession angle. Note that the longitude of perihelion is determined by the precession angle (as the planet precesses, its line of equinoxes changes), so our results don't depend on the longitude of perihelion. One area of concern could be that Pluto's period of precession

is in resonance with its orbital period. Dobrovolskis et al. show that the angle between Pluto's perihelion and its vernal equinox have a period of about three million Earth years, or about 12,000 Pluto years [8]. Although this period is slightly faster than Earth's precession, it is large enough so that Pluto's precession is negligible in a Pluto year. Thus, we should have no influence from the precession angle (or longitude of precession) in the calculations of Pluto's insolation.

It is important to note that because the mean annual insolation is symmetric across the equator, we have that any multi-year average must also be symmetric about the planet's equator.

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