Some Metric Properties of Attractors with Applications to Computer Simulations of Dynamical Systems

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One of the basic tasks in dynamical systems theory is to study structures which persists under small perturbations of the system. As an illustration of this idea, consider the concept of "structural stability". Roughly speaking, an appropriate topology is placed on the space of all systems under consideration. A given system is called "structurally stable" if it has a neighborhood in the space such that every system within that neighborhood is topologically conjugate to the given system. Besides being an elegant and interesting way to view dynamical systems, this concept can be justified as important outside of mathematics for the following reason. Since it is impossible to know a system exactly, it is imprudent to concentrate on those properties of a system which are not shared by nearby systems. Since all of the topological properties of a structurally stable system are shared by its neighbors, such systems are important to study.

In many applications, one wishes to carry this concept a step further. A metric is sometimes a natural part of an application, and the "distance" between two different systems can actually be assigned a meaningful quantity. Furthermore, one might have an estimate on how well the system is known, that is, one might be confident that the actual system under study is within a certain distance, say 2^{-64}, of the mathematical model. Although proving that the model system is structurally stable would be interesting, it would not necessarily imply any statement about the actual system. The model system might be structurally stable even though a topologically different system exists within a distance of 10^{-100} from it. A much more useful result would be that all systems within a distance 2^{-64} of the model system share its topological type, a result which makes a very strong statement about the actual system.

For applications of this sort, a concept of "ε-structurally stable" would be more useful. A system could be called "ε-structurally stable" if every system within an ε-neighborhood of the given system shared its topological type. The usual notion of structural stability could then be stated thus: a system is called "structurally stable" if there exists an ε such that it is ε-structurally stable. The difference between the two concepts is that, in one case, ε is known only to exist, while in the other case, an actual estimate for ε is known.

The discussion so far has centered around the notion of structural stability as an illustration of the difference between knowing the value of ε and knowing the existence of ε. However, structural stability is an extremely strong demand on a system, since it insists on the persistence of every fine topological detail. Sometimes the study of persistence of much coarser properties is sufficient for the particular application. One of the coarsest of properties is the existence of an attractor, and this is the property under investigation here. Again speaking roughly, an attractor is an invariant set which attracts everything in a neighborhood of itself. Attractors persist in the sense that, given a system with an attractor, every nearby system has a corresponding attractor. As before, one can ask how big a perturbation is allowed. As described in detail below, the answer is intimately tied to the strength by which the attractor attracts. Strong attractors persist further than weak attractors. Indeed, in Section 6 below is introduced a notion of the strength of an attractor called the "intensity" of attraction. An attractor will persist for all systems within a distance, in the C^0 metric, given by the intensity of attraction.

The motivating consideration for this study, and its only application in this paper, is computer simulation of dynamical systems. In a sense, this paper can be viewed as a study of the effect of round-off error on the problem of using direct computer simulations to find attractors of maps. For example, suppose that a map on some Euclidean space is being studied via simulations on a computer, and suppose that the round-off error introduced by the computer is of the order of 2^{-64}. If the system has an attractor which persists under perturbations of size greater than 2^{-64}, then there is some chance that the computer will find the attractor despite the introduced round-off error.

On the other hand, an attractor which fails to persist for some perturbation of size less than 2^{-64} might easily be missed by the computer simulation.

An interesting by-product of this study appears in Section 5, where a proof of the existence of an attractor block is given. An "attractor block" for a given attractor is a compact neighborhood of the attractor which maps strictly interior to itself and which converges to the attractor under iteration of the map. An "ε-pseudo-orbit" is a sequence of points obtained by successively following the map and then jumping a distance less then ε. It turns out that, for small enough ε, the set of all points accessible by ε-pseudo-orbits starting on a given attractor is an attractor block for that attractor.

The setting for this study is the iteration of maps on locally compact metric spaces. The notions introduced here, in particular, the notion of the "intensity" of an attractor, are inherently metric, not topological, in character. One may just as well think of maps on a Euclidean space or on a manifold, but, since no structure other than the metric is needed here, none is assumed. About the only advantage of not assuming compactness is the direct applicability, without further comment, to Euclidean spaces. No new insights are needed to prove the theorems in this more general setting of locally compact spaces; only care is needed in the definitions. An important step, not addressed in this paper, would be to extend the results to spaces which are not locally compact, such as infinite dimensional function spaces, and to thereby achieve some applicability to systems defined by partial differential equations or by differential-delay equations. Another direction, actively being pursued by Easton [**5**] and by Norton [**11**] is to study the entire Conley decomposition in these same metric terms.

Two of the examples worked out in the last section below illustrate cases where the intensity of attraction is so low that one cannot expect to find the attractor by computer simulation. In both cases, the attractor is a periodic orbit with a fairly low period. Both examples occur in systems which have been studied intensively by computer simulations. When viewed in one light, these examples illustrate that the computer might not be the best tool for uncovering certain phenomena considered important in the mathematical theory of dynamical systems. When viewed in a different light, these examples illustrate that certain phenomena considered important in the mathematical theory of dynamical systems may be of little importance in applications. Which light ultimately outshines the other remains to be seen. What is clear is that a great deal of further investigation is needed to illuminate the connections between mathematical experimentation and mathematical theory.

1 Iteration of Maps on Locally Compact Metric Spaces

Throughout this paper, (X, d) denotes a locally compact metric space, while ϕ denotes a continuous map from X to itself. The following standard definitions and notations are used.

If S is a subset of X, then a *neighborhood* of S is a set U containing S in its interior. That is, there exists an open set V such that $S \subset V \subset U$. If U is itself open, it is called an *open neighborhood* of S. If U is compact, it is called a *compact neighborhood* of S.

The closure of S is denoted by $\overline{S}$, the interior of S by S^o, and the complement of S by S^c. The relative complement of S_2 with respect to S_1, that is, the set $S_1 \cap S_2^c$, is denoted by $S_1 \setminus S_2$.

The following standard result states that every compact set has arbitrarily small compact neighborhoods.

Lemma 1.1 *If K is compact and if V is a neighborhood of K, then there exists a compact neighborhood G of K such that $G \subset V$.*

The notion of convergence in the "Hausdorff metric" will be used implicitly throughout this paper. However, since the concept will not be used in its full generality, the following notation is introduced for the special case of interest here.

Let S_n, $n \geq 0$, be a sequence of subsets of X, and let $S \subset X$. The notation

$$S_n \searrow S, \text{ as } n \to \infty$$

means that

(1) $k > n \Rightarrow S_k \subset S_n$,

(2) $S = \bigcap_{n \geq 0} S_n$, and

(3) $\forall$ neighborhood V of S, $\exists\, m \geq 0$, such that $S_m \subset V$.

Recall that, if S_n is compact for every n, then so is S. If, in addition, each S_n is nonempty, then so is S.

The following standard results will be used below.

Lemma 1.2 *If K_n is a nested sequence of compact sets, then*

$$K_n \searrow \bigcap_{n\geq 0} K_n, \text{ as } n \to \infty.$$

Lemma 1.3 *If S_n is a nested sequence of subsets of X, if K is a compact set satisfying $K \subset S_n$, for all $n \geq 0$, and if for every compact neighborhood G of K there exists an $m \geq 0$ such that $S_m \subset G$, then $S_n \searrow K$, as $n \to \infty$.*

Lemma 1.4 *If $S_n \searrow K$, as $n \to \infty$, if K is compact, and if $\phi : X \to X$ is continuous, then $\phi(S_n) \searrow \phi(K)$, as $n \to \infty$.*

If S_ε, $\varepsilon > 0$, is a family of subsets of X, then an analogous definition can be given for the notation

$$S_\varepsilon \searrow S, \text{ as } \varepsilon \to 0+.$$

Results analogous to those of Lemmas 1.2, 1.3, and 1.4 also hold.

An important concept in the study of the dynamics of the iterations of the coninuous map $\phi : X \to X$ is that of "invariance". The following definition is standard.

Definition: A set S is called *positively invariant* if $\phi(S) \subset S$. It is called *invariant* if $\phi(S) = S$.

Note that if S is positively invariant, so is $\overline{S}$. The following question arises. Does the invariance of S imply the invariance of $\overline{S}$? If ϕ is a homeomorphism, then the answer is "yes". The answer is also "yes" if X is compact or if $\overline{S}$ is compact. However, the following example shows that the answer in general is "no".

Example 1.5 *Let X be the real number line, let $\phi(x) = e^{x \sin x}$, and let $S = (0, \infty)$. Note that $\phi(S) = S$ but that $\phi(\overline{S}) = S \neq \overline{S}$.*

The "omega limit set" of a subset of X, a notion discussed extensively by Conley in the case of flows, is another important concept. For $S \subset X$, denote

$$\tau_n(S) \equiv \bigcup_{k\geq n} \phi^k(S),$$

$$\omega(S) \equiv \bigcap_{n\geq 0} \overline{\tau_n(S)}.$$

The set $\omega(S)$ is called the *omega limit set* of S. Note that $\omega(S)$ is closed. If S consists of a single point, then the above definition reduces to the classical definition of the omega limit set of a point:

$$\omega(x) \equiv \omega(\{x\}).$$

The following lemma establishes some elementary properties of the sequence $\tau_n(S)$.

Lemma 1.6 *The following properties hold whenever they are defined.*
(1) $\phi(\tau_n(S)) = \tau_{n+1}(S) \subset \tau_n(S)$.
(2) $\overline{\phi(\overline{\tau_n(S)})} = \overline{\tau_{n+1}(S)}$.
(3) $\phi(S) \subset S \Rightarrow \tau_n(S) = \phi^n(S)$.

Proof: Properties (1) and (3) follow immediately from the definition. To prove property (2), note that property (1) implies that $\tau_{n+1}(S) = \phi(\tau_n(S)) \subset \phi(\overline{\tau_n(S)}) \subset \overline{\phi(\tau_n(S))} = \overline{\tau_{n+1}(S)}$. Application of the closure operation to this formula yields property (2), and the proof is complete.

The following two lemmas establish results to be used in later sections. Both give sufficient conditions for $\overline{\tau_n(S)}$ to converge in the Hausdorff metric.

Lemma 1.7 *If S is nonempty, and if $\overline{\tau_0(S)}$ is compact, then $\omega(S)$ is a nonempty compact set and $\overline{\tau_n(S)} \searrow \omega(S)$.*

Proof: Since $\overline{\tau_0(S)}$ is compact, property (1) of Lemma 1.6 implies that $\overline{\tau_n(S)}$ is a nested sequence of nonempty compact sets. Therefore $\omega(S)$ is nonempty and compact. Application of Lemma 1.2 completes the proof.

Lemma 1.8 *If G is a nonempty compact positively invariant set, then $\omega(G)$ is a nonempty compact set and $\phi^n(G) \searrow \omega(G)$.*

Proof: Since G is positively invariant, property (3) of Lemma 1.6 implies that $\overline{\phi^n(G)} = \overline{\tau_n(G)}$. Since G is compact, so is $\phi^n(G)$, and, therefore, $\phi^n(G) = \overline{\tau_n(G)}$. Application of Lemma 1.7 completes the proof.

The following lemma establishes some elementary properties of the omega limit set.

Lemma 1.9 *The following properties hold whenever they are defined.*
(1) $S_1 \subset S_2 \Rightarrow \omega(S_1) \subset \omega(S_2)$.
(2) $\phi(S) = S \Rightarrow \omega(S) = \overline{S}$.
(3) $\phi(S) \subset S \Rightarrow \omega(S) = \bigcap_{n\geq 0} \overline{\phi^n(S)}$.
(4) $\phi(S) \subset S \Rightarrow \omega(S) \subset \overline{S}$.
(5) $\omega(\phi(S)) = \omega(S)$.
(6) $\phi(\omega(S)) \subset \omega(S)$.

Proof: The proofs of properties (1) and (2) follow immediately from the definitions. The proof of property (3) follows from the definition and from property (3) of Lemma 1.6. To prove property (4), note that $\phi(S) \subset S$ implies that $\phi^n(S) \subset S$, for all $n \geq 0$, which implies that $\overline{\phi^n(S)} \subset \overline{S}$, for all $n \geq 0$. The result follows by combining this last inclusion with property (3). The proof of property (5) follows immediately fom the definitions. To prove property (6), note that $\phi(\omega(S)) = \phi(\bigcap_{n\geq 0} \overline{\tau_n(S)}) \subset \bigcap_{n\geq 0} \phi(\overline{\tau_n(S)}) \subset \bigcap_{n\geq 0} \overline{\phi(\tau^n(S))}$. Property (1) of Lemma 1.6 therefore implies that $\phi(\omega(S)) \subset \bigcap_{n\geq 0} \overline{\tau_{n+1}(S)} = \omega(S)$. The proof is complete.

Property (6) of Lemma 1.9 states that $\omega(S)$ is positively invariant, and it raises the question of whether $\omega(S)$ is invariant. Example 1.5 above shows that, in general, the answer is "no", since $\omega(S) = \overline{S}$ is not invariant for that example. On the other hand, if ϕ is a homeomorphism, it is easy to show that the answer is "yes". The following lemma states other conditions for which $\omega(S)$ is invariant.

Lemma 1.10 *If S is nonempty, and if any one of the following conditions holds, then $\omega(S)$ is a nonempty compact invariant set.*
(1) $\overline{\tau_0(S)}$ *is compact.*
(2) S *is compact and positively invariant.*
(3) X *is compact.*

Proof: To establish the sufficiency of condition (1), note that, since $\overline{\tau_0(S)}$ is compact, Lemma 1.7 implies that $\omega(S)$ is nonempty and compact and that $\overline{\tau_n(S)} \searrow \omega(S)$. Lemma 1.4 then implies that $\phi(\overline{\tau_n(S)}) \searrow \phi(\omega(S))$, as $n \to \infty$. Since $\overline{\tau_n(S)}$ is compact, so is $\phi(\overline{\tau_n(S)})$. Property (2) of Lemma 1.6 therefore implies that $\overline{\tau_{n+1}(S)} = \phi(\overline{\tau_n(S)})$ and hence that $\overline{\tau_{n+1}(S)} \searrow \phi(\omega(S))$, as $n \to \infty$. Therefore $\phi(\omega(S)) = \omega(S)$, that is, $\omega(S)$ is invariant. Next note that conditon (2) implies condition (1), since, if S is compact and positively invariant, then $\overline{\tau_0(S)} = S$. Finally, note that condition (3) implies condition (1), since a closed subset of a compact set is compact. The proof is complete.

It is tempting to speculate that the compactness of $\omega(S)$ is sufficient to insure its invariance. However, the following example shows that the speculation would be incorrect.

Example 1.11 *Let*

$$X = \{(x,k) : x \in \mathbf{R}, k \in \mathbf{Z}, k \geq 0\},$$

$$\phi(x,k) = \begin{cases} (x+1, k-1), & k \geq 2, \\ (\sin x, 0), & k = 1, \\ (x/2, 0), & k = 0, \end{cases}$$

and let

$$K = \{(x,0) : -1 \leq x \leq 1\}, \;\; S = \{(x,k) : x \geq 0, k \geq 1\}.$$

Note that $\omega(S) = K$, which is compact but not invariant.

2 Attractors

The following definition is a modification to this setting of that given by Conley [**2,3**] for flows on compact metric spaces.

Definition: A set A is an *attractor* for ϕ if

(1) A is a nonempty compact invariant set and

(2) there exists a neighborhood U of A such that $\omega(U) = A$.

Some authors would call A an "attracting set" and would reserve the name "attractor" for an attracting set with further properties. However, in this paper Conley's terminology will be followed.

The preceeding definition is somewhat weak in the sense that the only assumption on the neighborhood U is that $\omega(U) = A$. However, this assumption is actually very strong. For example, the neighborhood U can be taken to be compact, positively invariant, and arbitrarily close to A, as stated in Theorem 2.1 below. Also, the notion of attractor corresponds to the more classical notion of "asymptotically stable". The following definition is a translation to this setting of the classical definition as found in the book by LaSalle and Lefschetz [**9**]

Definition: A compact invariant set S is called *stable* if for each neighborhood V of S there exists a neighborhood V' of S such that

$$\phi^n(V') \subset V, \forall n \geq 0.$$

The set is called *asymptotically stable* if it is stable and if there exists a neighborhood W of S such that $\omega(x) \subset S, \forall x \in W$.

Theorem 2.1 *If A is a nonempty compact invariant set, then the following statements are equivalent.*

(1) *A is an attractor.*

(2) *If V is any neighborhood of A, then there exists a compact neighborhood K of A such that $K \subset V$ and $\omega(K) = A$.*

(3) *If V is any neighborhood of A, then there exists a positively invariant compact neighborhood G of A such that $G \subset V$ and $\omega(G) = A$.*

(4) *A is asymptotically stable.*

Proof that (1) $\Rightarrow$(2): Since A is an attractor, there exists a neighborhood U of A such that $\omega(U) = A$. Lemma 1.1 implies the existence of a compact neighborhood K of A such that $K \subset U \cap V$. Properties (1) and (2) of Lemma 1.9 then imply that $A = \omega(A) \subset \omega(K) \subset \omega(U) = A$. Therefore, $\omega(K) = A$, and the proof is complete.

Proof that (2)$\Rightarrow$(3): Let V be a neighborhood of A and let K be a compact neighborhood of A such that $K \subset V$ and $\omega(K) = A$. Since ϕ is continuous and A is invariant, we can find an open neighborhood S of A such that

$$S \subset K \text{ and } \phi(S) \subset K.$$

Let $K_n = \overline{\tau_n(K)}$. Since $A = \omega(K) = \bigcap_{n \geq 0} K_n$, we have that $\{K_n^c\}$ is an open cover of $K \setminus S$. Since $K \setminus S$ is compact, there is a finite subcover. Since $\{K_n^c\}$ is nested, there exists an $m \geq 0$ such that $K_m^c \supset K \setminus S$, and hence such that $K_n \subset K^c \cup S$, for all $n \geq m$. Since $\phi^n(K) \subset \tau_n(K) \subset K_n$, it follows that $\phi^n(K) \subset K^c \cup S$, for all $n \geq m$. Indeed, for any $Z \subset K$,

$$\phi^n(Z) \subset K^c \cup S, \forall n \geq m.$$

In other words,

$$\phi^n(Z) \subset K \Rightarrow \phi^n(Z) \subset S, \forall n \geq m.$$

Therefore,

$$\phi^n(Z) \subset K \Rightarrow \phi^{n+1}(Z) \subset \phi(S) \subset K, \forall n \geq m.$$

Induction applied to the preceeding formula yields

(2-1) $$Z \subset K \text{ and } \phi^m(Z) \subset K \Rightarrow \phi^n(Z) \subset K, \forall n \geq m.$$

Now let

$$S' \equiv S \cap \phi^{-1}(S) \cap \cdots \cap \phi^{-(m-1)}(S).$$

Note that S' is a neighborhood of A and that

$$\phi^n(S') \subset S \subset K, \text{ for } n = 0, 1, \ldots, m-1.$$

Since $\phi(S) \subset K$, it follows that $\phi^n(S') \subset K$ for $n = m$ as well. Formula (2-1) with $Z = S'$ implies that $\phi^n(S') \subset K$, for all $n \geq m$. Therefore,

$$\phi^n(S') \subset K, \forall n \geq 0,$$

from which it follows that $\tau_0(S') \subset K$. Now let $G = \overline{\tau_0(S')}$. Note that $A \subset S' \subset G \subset K \subset V$. Therefore, G is a compact neighborhood of A satisfying $\underline{G \subset V}$. Properties (1) and (2) of Lemma 1.6 imply that $\phi(\overline{\tau_0(S')}) \subset \overline{\tau_1(S')} \subset \overline{\tau_0(S')}$ and hence that $\phi(G) \subset G$. Therefore, G is positively invariant, and the proof is complete.

Proof that (3) $\Rightarrow$ (4): Let V be a neighborhood of A, and let G be a positively invariant compact neighborhood of A satisfying $G \subset V$ and $\omega(G) = A$. Let $W = V' = G$. Since G is positively invariant, $\phi^n(V') \subset G \subset V$, for all $n \geq 0$. Therefore A is stable. Furthermore, for all $x \in W = G$, property (1) of Lemma 1.9 implies that $\omega(x) \subset \omega(G) = A$. Therefore, A is asymptotically stable, and the proof is complete.

Proof that (4) $\Rightarrow$ (1): By definition, there exists a neighborhood W of A such that, $\forall x \in W$, $\omega(x) \subset A$. Since A is stable, there exists a neighborhood W' of A such that $\phi^n(W') \subset W$, $\forall n \geq 0$. Without loss of generality, W and W' can be assumed to be compact. Now let V be any neighborhood of A. It will be shown that $\omega(W') \subset \overline{V}$. Since V is arbitrary and since A is closed, it follows that $\omega(W') \subset A$, and hence that A is an attractor.

Since A is stable, there exists an open neighborhood V' of A such that $\phi^n(V') \subset V$, $\forall n \geq 0$. Since $\phi^n(W') \subset W$, $\forall n \geq 0$, it follows that, for each

$x \in W'$, $\phi^n(x) \subset W$, $\forall n \geq 0$, and hence that $\tau_0(x) \subset W$, $\forall x \in W'$. Since W is compact, $\overline{\tau_0(x)}$ is compact, and Lemma 1.7 implies that $\overline{\tau_n(x)} \searrow \omega(x) \subset A$. Therefore, there exists an n_x such that $\overline{\tau_{n_x}(x)} \subset V'$, which implies that $\phi^{n_x}(x) \in V'$. Since ϕ^{n_x} is continuous and since V' is open, there exists a neighborhood U_x of x such that $\phi^{n_x}(U_x) \subset V'$. Since $\phi^k(V') \subset V$, $\forall k \geq 0$, it follows that

$$\phi^n(U_x) \subset V, \forall n \geq n_x.$$

Since W' is compact and since $\{U_x\}$ covers W', there exists a finite subcover $U_{x_1}, U_{x_2}, \ldots, U_{x_m}$. If $N = \max(n_{x_1}, n_{x_2}, \ldots, n_{x_m})$, then $\phi^n(W') \subset V$, $\forall n \geq N$, and hence $\tau_n(W') \subset V$, $\forall n \geq N$. Therefore, $\omega(W') \subset \overline{V}$, and the proof is complete.

The set of all points attracted to A is called the *domain of attraction of A*, denoted

$$\mathcal{D}(A) \equiv \{x \in X : \emptyset \neq \omega(x) \subset A\}.$$

Note that, if X is compact, it is always true that $\omega(x) \neq \emptyset$. This condition is imposed in the noncompact case to avoid including points which escape to infinity.

The following theorems establish that $\mathcal{D}(A)$ is an open set and that every compact subset K of $\mathcal{D}(A)$ satisfies $\omega(K) \subset A$.

Theorem 2.2 *If G is a compact positively invariant neighborhood of an attractor A such that $\omega(G) = A$, then*

$$G \subset \mathcal{D}(A) = \bigcup_{n \geq 0} \phi^{-n}(G^0),$$

and hence $\mathcal{D}(A)$ is open.

Proof: Let $x \in G$. Property (1) of Lemma 1.9 implies that $\omega(x) \subset \omega(G) = A$. Since G is positively invariant, $\phi^n(x) \in G$, $\forall n \geq 0$, which implies that $\tau_0(x) \subset G$. Since G is compact, so is $\overline{\tau_0(x)}$. Therefore, Lemma 1.7 implies that $\omega(x) \neq \emptyset$, and the inclusion $G \subset \mathcal{D}(A)$ has been established.

Now let $x \in \mathcal{D}(A)$, and suppose that $\phi^n(x) \notin G^0$, $\forall n \geq 0$. Then $\phi^n(x) \in (G^0)^c$, $\forall n \geq 0$, which implies that $\omega(x) \subset (G^0)^c$, which in turn implies that $\omega(x) \cap A = \emptyset$, which is a contradiction. Therefore, $x \in \phi^{-n}(G^0)$, for some $n \geq 0$, and the inclusion $\mathcal{D}(A) \subset \bigcup_{n \geq 0} \phi^{-n}(G^0)$ has been established.

Finally, let $x \in \phi^{-n}(G^0)$, for some $n \geq 0$. Then $\phi^n(x) \subset G^0 \subset G \subset \mathcal{D}(A)$, which, together with property (5) of Lemma 1.9 implies that $\emptyset \neq \omega(\phi^n(x)) = \omega(x) \subset A$. Therefore, $x \in \mathcal{D}(A)$, which establishes the final inclusion and completes the proof.

Theorem 2.3 *If A is an attractor and if K is a compact subset of $\mathcal{D}(A)$, then $\omega(K) \subset A$.*

Proof: Statement (3) of Theorem 2.1 implies the existence of a positively invariant compact neighborhood G of A such that $\omega(G) = A$. Theorem 2.2 implies that $K \subset \mathcal{D}(A) = \bigcup_{n\geq 0} \phi^{-n}(G^0)$. Since K is compact, there exists a finite subcover. That is, $\exists m \geq 0$ such that

$$K \subset \bigcup_{n=0}^{m} \phi^{-n}(G^0).$$

In other words, for each $x \in K$, there is an $n \in [0, m]$ such that $\phi^n(x) \in G^0$. Since G is positively invariant,

$$0 \leq n \leq m \text{ and } \phi^n(x) \subset G^0 \Rightarrow \phi^m(x) \subset G.$$

Therefore, $\phi^m(K) \subset G$. Properties (1) and (5) of Lemma 1.9 imply that $\omega(K) = \omega(\phi^m(K)) \subset \omega(G) = A$, which completes the proof.

3 Attractor Blocks

Conley and Easton [**4**] introduced the concept of an ”isolating block” as a tool for the study of the topological properties of isolated invariant sets. A special case is an isolating block for an attractor, also called an ”attractor block”. A set is an attractor block if its image is strictly interior to itself.

Definition: A set B is called an *attractor block* for ϕ if B is compact and nonempty and if $\phi(B) \subset B^o$.

The following two theorems give the correspondence between attractors and attractor blocks. Every attractor block has an attractor in its interior given by allowing the block to converge through successive iterates of the map. The attractor thus obtained is the maximal invariant set inside the block. Conversely, every attractor can be surrounded by an attractor block with the property that the attractor is the maximal invariant set inside the block.

Theorem 3.1 *If B is an attractor block, then $A \equiv \omega(B)$ is an attractor.*

Theorem 3.2 *If A is an attractor and if V is any neighborhood of A, then there exists an attractor block $B \subset V$ such that $A = \omega(B)$.*

If B is an attractor block, then $\omega(B)$ is called "the attractor associated with B". If A is an attractor and if B is an attractor block such that $\omega(B) = A$, then B is called "an attractor block associated with A". Of course, although the attractor block uniquely determines its associated attractor, many distinct attractor blocks can be found associated with a given attractor.

An attractor block has stronger stability properties than those of the attractor. The attractor itself may change dramatically under perturbation of the system, but the attractor block remains an attractor block under perturbation. This stability is exploited by Conley in his study of the Morse index and will be used extensively below in the present investigation.

Theorem 3.1 is proved next. The proof of Theorem 3.2 is postponed until after the discussion of "pseudo-orbits", where it becomes a consequence of Theorem 4.6.

Proof of Theorem 3.1: Since B is compact, nonempty and positively invariant, condition (2) of Lemma 1.10 implies that $A \equiv \omega(B)$ is compact, nonempty and invariant. Since $A \subset B$, it follows that $A = \phi(A) \subset \phi(B) \subset B^o \subset B$. Therefore, B is a neighborhood of A, which implies that A is an attractor and completes the proof.

The remainder of this section is devoted to a metric characterization of attractor blocks. Some notation is introduced to facilitate the description.

The set of points less than a positive distance ε from a point x is denoted

$$N_\varepsilon(x) \equiv \{y : d(x, y) < \varepsilon\}.$$

The set of points whose minimum distance from a subset S is less than ε is denoted

$$N_\varepsilon(S) \equiv \bigcup_{x \in S} N_\varepsilon(x).$$

Note that $N_\varepsilon(S)$ is an open neighborhood of S.

The following standard result will be used below. It is stated here without proof.

Lemma 3.3 *If V is a neighborhood of a compact set K, then there exists an $\varepsilon > 0$ such that $N_\varepsilon(K) \subset V$.*

Some elementary properties of $N_\varepsilon(S)$ are gathered into the following lemma. The proofs are easy and will be omitted.

Lemma 3.4 *The following properties hold whenever they are defined.*
(1) $S_1 \subset S_2 \Rightarrow N_\varepsilon(S_1) \subset N_\varepsilon(S_2)$.
(2) $\varepsilon < \delta \Rightarrow N_\varepsilon(S) \subset N_\delta(S)$.
(3) $N_\varepsilon(\bigcup_{\alpha \in I} S_\alpha) = \bigcup_{\alpha \in I} N_\varepsilon(S_\alpha)$, *for any index set* I.
(4) $N_\varepsilon(\overline{S}) = N_\varepsilon(S)$.

The following lemma will be used below.

Lemma 3.5 *If $S_\varepsilon \searrow K$, as $\varepsilon \to 0+$, and if K is compact, then $N_\varepsilon(S_\varepsilon) \searrow K$, as $\varepsilon \to 0+$.*

Proof: Properties (1) and (2) of Lemma 3.4 imply that $N_\varepsilon(S_\varepsilon)$ is a nested family. Furthermore, $K \subset S_\varepsilon \subset N_\varepsilon(S_\varepsilon)$, for all $\varepsilon > 0$. Let G be a compact neighborhood of K. In view of Lemma 1.3, the proof will be complete once it is established that $N_\delta(S_\delta) \subset G$, for some $\delta > 0$.

Let G' be a compact neighborhood of K such that $G' \subset G^o$. The definition of $S_\varepsilon \searrow K$ implies the existence of a $\delta > 0$ such that $S_\delta \subset G'$. Lemma 3.3 implies that δ can be chosen so that $N_\delta(G') \subset G$ as well. Property (1) of Lemma 3.4 now implies that $N_\delta(S_\delta) \subset G$, and the proof is complete.

A standard notation has already been used without comment, namely, if $\phi : X \to X$, then, for $S \subset X$,

$$\phi(S) \equiv \{\phi(x) : x \in S\}.$$

In other words, the map $\phi : X \to X$ induces a map $\phi : 2^X \to 2^X$, where 2^X denotes the set of all subsets of X. Another map

$$\phi_\varepsilon : 2^X \to 2^X$$

can be defined by

$$\phi_\varepsilon(S) \equiv N_\varepsilon(\phi(S)), \text{ for } \varepsilon > 0.$$

Note that $\phi_\varepsilon(S)$ is the set of all points within a distance less than ε of the image of S under ϕ. It will be convenient to adopt the notational convention

$$\phi_\varepsilon(S) \equiv \emptyset \text{ , for } \varepsilon \leq 0.$$

Lemma 3.6 *The following properties hold whenever they are defined.*
(1) $S_1 \subset S_2 \Rightarrow \phi_\varepsilon(S_1) \subset \phi_\varepsilon(S_2)$.
(2) $\varepsilon < \delta \to \phi_\varepsilon(S) \subset \phi_\delta(S)$.
(3) $\phi_\varepsilon(\bigcup_{\alpha \in I} S_\alpha) = \bigcup_{\alpha \in I} \phi_\varepsilon(S_\alpha)$, *for any index set* I.
(4) $\phi_\varepsilon(\overline{S}) = \phi_\varepsilon(S)$.

Proof: Properties (1)-(3) are consequences of properties (1)-(3) of Lemma 3.4 and the analogous properties for the map ϕ. Property (1) of Lemma 3.4 implies that $\phi_\varepsilon(S) \subset \phi_\varepsilon(\overline{S})$. Since $\phi(\overline{S}) \subset \overline{\phi(S)}$, properties (1) and (4) of Lemma 3.4 imply that $\phi_\varepsilon(\overline{S}) = N_\varepsilon(\phi(\overline{S})) \subset N_\varepsilon(\overline{\phi(S)}) = N_\varepsilon(\phi(S)) = \phi_\varepsilon(S)$, which establishes property (4) and completes the proof.

Lemma 3.7 *If* $S_\varepsilon \searrow K$, *as* $\varepsilon \to 0+$, *and if* K *is compact, then*

$$\phi_\varepsilon(S_\varepsilon) \searrow \phi(K) \text{ , as } \varepsilon \to 0+ .$$

Proof: Lemma 1.4 implies that $\phi(S_\varepsilon) \searrow \phi(K)$. Since $\phi(K)$ is compact, Lemma 3.5 implies that $\phi_\varepsilon(S_\varepsilon) = N_\varepsilon(\phi(S_\varepsilon)) \searrow \phi(K)$, and the proof is complete.

Since ϕ_ε is a map, it can be iterated. The standard notation for the iterates will be used, namely,

$$\phi_\varepsilon^0(S) \equiv S,$$

$$\phi_\varepsilon^n(S) \equiv \phi_\varepsilon(\phi_\varepsilon^{n-1}(S)) \text{ , for } n \geq 1.$$

The following two lemmas will be needed for the discussion of pseudo-orbits. The first follows readily from the corresponding properties of Lemma 3.6. The proof of the second is a simple argument using induction on n and Lemma 3.7.

Lemma 3.8 *The following properties hold whenever they are defined.*
(1) $S_1 \subset S_2 \Rightarrow \phi_\varepsilon^n(S_1) \subset \phi_\varepsilon^n(S_2)$.
(2) $\varepsilon < \delta \Rightarrow \phi_\varepsilon^n(S) \subset \phi_\delta^n(S)$.

Lemma 3.9 *If* K *is compact and if* $n \geq 0$, *then* $\phi_\varepsilon^n(K) \searrow \phi^n(K)$, *as* $\varepsilon \to 0+$.

It will be useful to measure the distance that an attractor block maps inside itself. For $S \subset X$, define

$$\beta(S) \equiv \sup\{\varepsilon : \phi_\varepsilon(S) \subset S\}.$$

Note that $\beta(S)$ is the minimum distance from $\phi(S)$ to S^c and that $\beta(S) \geq 0$, for any subset S.

Lemma 3.10 $\beta(\overline{S}) \geq \beta(S)$.

Proof: If $\phi_\varepsilon(S) \subset S$, then property (4) of Lemma 3.6 implies that $\phi_\varepsilon(\overline{S}) = \phi_\varepsilon(S) \subset S \subset \overline{S}$. In other words,

$$\{\varepsilon : \phi_\varepsilon(S) \subset S\} \subset \{\varepsilon : \phi_\varepsilon(\overline{S}) \subset \overline{S}\}.$$

The result follows from this inclusion, and the proof is complete.

Theorem 3.11 *A nonempty compact set K is an attractor block if and only if $\beta(K) > 0$.*

Proof: If $\beta(K) > 0$, then there exists an $\varepsilon > 0$ such that $\phi_\varepsilon(K) \subset K$. Since $\phi_\varepsilon(K)$ is an open set containing $\phi(K)$, it follows that $\phi(K) \subset K^o$ and hence that K is an attractor block. To establish the converse, assume that K is an attractor block and hence that $\phi(K) \subset K^o$. Lemma 3.7 implies that $\phi_\varepsilon(K) \searrow K$ and hence that $\phi_\delta(K) \subset K^o \subset K$, for some $\delta > 0$. Therefore, $\beta(K) \geq \delta > 0$, and the proof is complete.

The following corollary gives a sufficient condition on a subset for its closure to be an attractor block. The proof is an immediate consequence of Theorem 3.11 and Lemma 3.10.

Corollary 3.12 *Let S be a nonempty subset of X. If $\beta(S) > 0$ and if $\overline{S}$is compact, then $\overline{S}$ is an attractor block.*

4 Pseudo-orbits

The notion of a "pseudo-orbit" has had important applications in several different areas of dynamical systems. Most important has been the concept of "shadowing", which has been used to prove the existence of orbits corresponding to symbol shifts. Hammel, Yorke and Grebogi [**7,8**] have exploited extensively the fact that a pseudo-orbit is the actual object computed by a computer. They are able to show rigorously that certain orbits found by simulation correspond to real orbits for the original system.

Roughly speaking, an ε-pseudo orbit is obtained by successively following the system, each time making an "error" of size less than ε.

Definition: A sequence $(z_0, z_1, \ldots, z_n)$ of points in X satisfying

$$d(\phi(z_{k-1}), z_k) < \varepsilon \text{ , for } k = 1, 2, \ldots, n,$$

is called an ε-pseudo-orbit of length n.

Note that the definition of a real orbit can be written in an analogous way as follows. An orbit of length n is a sequence $(x_0, x_1, \ldots, x_n)$ such that

$$d(\phi(x_{k-1}), x_k) = 0 \text{ , for } k = 1, 2, \ldots, n.$$

Although Conley [**3**] uses the term "ε-chain" for an ε-pseudo-orbit, the latter terminology is more descriptive for the applications presented here and will be used throughout this paper.

It turns out that attractor blocks can be constructed from ε-pseudo-orbits. If one considers the set of all points which can be reached from an attractor A by an ε-pseudo-orbit, then, for sufficiently small ε, that set is an attractor block corresponding to A. This statement will be made precise and proved in this section.

The following notation will be used to denote the set of all ε-pseudo-orbits of length n starting in the set S.

$$\Psi_\varepsilon^n(S) = \{(z_0, z_1, \ldots, z_n) : z_0 \in S \text{ and } d(\phi(z_{k-1}), z_k) < \varepsilon \text{ for } 1 \leq k \leq n\}.$$

It will be convenient to have a notation for the kth coordinate of a pseudo-orbit. For $\mathbf{z} = (z_0, z_1, \ldots, z_n)$ and for $k = 0, 1, \ldots, n$, define

$$\pi_k \mathbf{z} \equiv z_k.$$

It is clear that ε-pseudo-orbits are closely related to the map ϕ_ε defined in the previous section. Indeed, an ε-pseudo-orbit is simply a sequence of points picked out of successive iterates of ϕ_ε. More precisely, $\mathbf{z}$ is an ε-pseudo-orbit if and only if

(4-1) $$\pi_k \mathbf{z} \in \phi_\varepsilon(\pi_{k-1}\mathbf{z}), \text{ for } k = 1, 2, \ldots, n.$$

Observe that the notation $\phi_\varepsilon(x) \equiv \phi_\varepsilon(\{x\})$ is used. The following lemma states that points in the kth iterate of S under ϕ_ε are precisely those points in the kth coordinate of some ε-pseudo-orbit staring in S.

Lemma 4.1 *Fix $k \geq 0$. For every $n \geq k$,*

$$\phi_\varepsilon^k(S) = \bigcup \{\{\pi_k \mathbf{z}\} : \mathbf{z} \in \Psi_\varepsilon^n(S)\}.$$

Proof: Denote the right hand side of the equation by S_k^n. The proof proceeds by induction on k. For $k = 0$, $\phi_\varepsilon^0(S) = S = S_0^n$. Assume that equality has been established for $k = j$ and for $n \geq j$, and assume that $n \geq j + 1$. Property (3) of Lemma 3.6 and the induction hypothesis imply that

(4-2) $$\phi_\varepsilon^{j+1}(S) = \phi_\varepsilon(\phi_\varepsilon^j(S)) = \phi_\varepsilon(S_j^n) = \bigcup\{\phi_\varepsilon(\pi_j \mathbf{z}) : \mathbf{z} \in \Psi_\varepsilon^n(S)\}.$$

Let $x \in S_{j+1}^n$. Then $x = \pi_{j+1}\mathbf{z}$, for some $\mathbf{z} \in \Psi_\varepsilon^n(S)$, which, when combined with formula (4-1), means that $x \in \phi_\varepsilon(\pi_j \mathbf{z})$. Equation (4-2) then implies that $x \in \phi_\varepsilon^{j+1}(S)$, and the inclusion

$$\phi_\varepsilon^{j+1}(S) \supset S_{j+1}^n$$

is established. Now let $x \in \phi_\varepsilon^{j+1}(S)$. Equation (4-2) implies that $x \in \phi_\varepsilon(\pi_j \mathbf{z})$, for some $\mathbf{z} \in \Psi_\varepsilon^n(S)$. Define $\mathbf{w} = (w_0, w_1, \ldots, w_n)$ by

$$w_i = \begin{cases} \pi_i \mathbf{z} & \text{for} \quad 0 \leq i \leq j, \\ x & \text{for} \quad i = j + 1, \\ \phi^{i-j-1}(x) & \text{for} \quad j + 1 < i \leq n. \end{cases}$$

Note that $\mathbf{w} \in \Psi_\varepsilon^n(S)$ and that $\pi_{j+1}\mathbf{w} = x$. Therefore, $x \in S_{j+1}^n$, and the inclusion

$$\phi_\varepsilon^{j+1}(S) \subset S_{j+1}^n$$

is established. The proof is complete.

The set of all points on all ε-pseudo-orbits of length n starting on the set S will be denoted

$$P_\varepsilon^n(S) \equiv \cup\{\{z_0, z_1, \ldots, z_n\} : (z_0, z_1, \ldots, z_n) \in \Psi_\varepsilon^n(S)\}.$$

Note that

$$P_\varepsilon^n(S) = \cup\{\{\pi_k \mathbf{z}\} : 0 \leq k \leq n \text{ and } \mathbf{z} \in \Psi_\varepsilon^n(S)\}.$$

The following lemma states that the set of all points on ε-pseudo-orbits starting on a set S is identical to the union of iterates of S under the map ϕ_ε.

Lemma 4.2 $P_\varepsilon^n(S) = \cup_{k=0}^n \phi_\varepsilon^k(S)$.

Proof: Lemma 4.1 and the previously noted formula imply that

$$P_\varepsilon^n(S) = \bigcup\{\{\pi_k \mathbf{z}\} : 0 \le k \le n \text{ and } \mathbf{z} \in \Psi_\varepsilon^n(S)\}$$

$$= \bigcup_{k=0}^{n} \bigcup\{\{\pi_k \mathbf{z}\} : \mathbf{z} \in \Psi_\varepsilon^n(S)\} = \bigcup_{k=0}^{n} \phi_\varepsilon^k(S),$$

which completes the proof.

The set of all points on all ε-pseudo-orbits of arbitrary length will be important in what is to follow. This set will be denoted

$$P_\varepsilon(S) \equiv \cup_{n=0}^\infty P_\varepsilon^n(S).$$

Some elementary properties of this set are collected in the following lemma.

Lemma 4.3 *The following properties hold whenever they are defined.*
(1) $P_\varepsilon(S) = \cup_{n=0}^\infty \phi_\varepsilon^n(S)$.
(2) $S \subset P_\varepsilon(S)$.
(3) $\varepsilon < \delta \Rightarrow P_\varepsilon(S) \subset P_\delta(S)$.
(4) $\beta(P_\varepsilon(S)) \ge \varepsilon$.

Proof: Property (1) is an immediate consequence of Lemma 4.2, while property (2) is an immediate consequence of the definition of $P_\varepsilon(S)$. Property (3) follows from property (1) and from property (2) of Lemma 3.8. Property (4) follows from the definition of $\beta$and from property (3) of Lemma 3.6, which implies that

$$\phi_\varepsilon(P_\varepsilon(S)) = \phi_\varepsilon(\cup_{n=0}^\infty \phi_\varepsilon^n(S)) = \cup_{n=0}^\infty \phi_\varepsilon^{n+1}(S) \subset P_\varepsilon(S).$$

The proof is complete.

Note that this last property implies that the set $P_\varepsilon(S)$ of all points accessible by ε-pseudo-orbits starting on S maps into itself by a distance at least ε. In view of Corollary 3.12, $\overline{P_\varepsilon(S)}$ would be an attractor block if it were compact.

Corollary 4.4 *If S is nonempty and if $\overline{P_\varepsilon(S)}$ is compact, then $\overline{P_\varepsilon(S)}$ is an attractor block.*

This property is exploited in the next lemma.

Lemma 4.5 *Let A be an attractor, let $\varepsilon > 0$, and define $B_\varepsilon \equiv \overline{P_\varepsilon(A)}$. If B_ε is compact and if $B_\varepsilon \subset \mathcal{D}(\mathcal{A})$, then B_ε is an attractor block associated with A.*

Proof: Corollary 4.4 implies that B_ε is an attractor block. Property (2) of Lemma 4.3 implies that $A \subset B_\varepsilon$, which, with properties (1) and (2) of Lemma 1.9, implies that $A = \omega(A) \subset \omega(B_\varepsilon)$. Theorem 2.3 implies that $\omega(B_\varepsilon) \subset A$. Therefore, $A = \omega(B_\varepsilon)$, and the proof is complete.

It remains to show that $P_\varepsilon(A)$ is close to A for small ε.

Theorem 4.6 *If A is an attractor, then $P_\varepsilon(A) \searrow A$, as $\varepsilon \to 0+$.*

Proof: Properties (2) and (3) of Lemma 4.3 imply that $P_\varepsilon(A)$ is a nested family and that $A \subset P_\varepsilon(A)$. Recall Lemma 1.3, and let G be an arbitrary compact neighborhood of A. The proof will be complete once it is established that $P_\delta(A) \subset G$, for some $\delta > 0$.

In view of statement (3) of Theorem 2.1, it can be assumed that G is positively invariant. Since Lemma 1.8 implies that $\phi^n(G) \searrow A$, as $n \to \infty$, an m can be chosen so that

$$\phi^m(G) \subset G^o.$$

Since $\phi^m(G)$ is compact, Lemma 3.9 implies that

$$\phi_\varepsilon^m(G) \searrow \phi^m(G), \text{ as } \varepsilon \to 0+.$$

Since A is compact and invariant, the same lemma implies that , for each fixed n,

$$\phi_\varepsilon^n(A) \searrow \phi^n(A) = A, \text{ as } \varepsilon \to 0+.$$

Since G is a neighborhood of both $\phi^m(G)$ and A, a $\delta > 0$ can be chosen so that

$$\phi_\delta^m(G) \subset G \text{ and}$$
$$\phi_\delta^n(A) \subset G, \text{ for } n = 0, 1, \ldots, m-1.$$

Note that, if $\phi_\delta^k(A) \subset G$, then property (1) of Lemma 3.8 implies that $\phi_\delta^{m+k}(A) = \phi_\delta^m(\phi_\delta^k(A)) \subset \phi_\delta^m(G) \subset G$. Therefore, by induction,

$$\phi_\delta^n(A) \subset G, \text{ for } n \geq 0.$$

Property (1) of Lemma 4.3 now implies that $P_\delta(A) \subset G$, and the proof is complete.

An immediate consequence of Theorem 4.6 is the following corollary.

Corollary 4.7 *If U is a neighborhood of an attractor A, then there exists an $\varepsilon > 0$ such that $\overline{P_\varepsilon(A)}$ is compact and is a subset of U.*

The machinery just developed now provides a proof of Theorem 3.2.

Proof of Theorem 3.2: Let A be an attractor, let V be a neighborhood of A, and define $B_\varepsilon \equiv \overline{P_\varepsilon(A)}$. The preceeding corollary implies the existence of an $\varepsilon > 0$ such that B_ε is compact and $B_\varepsilon \subset V \cap \mathcal{D}(\mathcal{A})$. Lemma 4.5 implies that B_ε is an attractor block associated with A, which completes the proof.

5 Intensity of Attraction

It is now possible to assign a precise quantity to measure the strength of attraction of an attractor. Two definitions are given, followed by a proof that the two are really the same. The first, called here the "intensity", assigns to an attractor A the supremum over all values of $\beta(B)$ such that B is an attractor block for A. That is, every attractor block B associated with A has the property that the minimum distance from the image of B to its complement does not exceed the intensity of A. Furthermore, the intensity is the smallest such number. The second definition, called here the "chain intensity", assigns to an attractor A the supremum over all values of ε such that every ε-pseudo-orbit starting in A stays in some compact subset of the domain of attraction of A. That is, every ε-pseudo-orbit which starts in A and for which ε does not exceed the chain intensity of A remains inside the domain of attraction of A. On the other hand, if ε does exceed the chain intensity of A, then one can find an ε-pseudo-orbit starting on A and leaving every compact subset of the domain of attraction of A.

Definition: Let A be an attractor. The *intensity* of A is defined as

$$\nu(A) \equiv \sup\{\beta(B) : B \text{ is an attractor block associated with } A\}.$$

The *chain intensity* of A is defined as

$$\mu(A) \equiv \sup\{\varepsilon : P_\varepsilon(A) \subset K \subset \mathcal{D}(A), \text{ where } K \text{ is compact }\}.$$

Note that Theorems 3.2 and 3.11 imply that $\nu(A)$ is positive. Note also that Corollary 4.7 implies the same for $\mu(A)$.

Theorem 5.1 $\nu(A) \equiv \mu(A)$.

Proof: Let $\delta < \nu(A)$. By definition, there is an attractor block B associated with A such that $\beta(B) > \delta$, which means that $\phi_\delta(B) \subset B$. Property (1) of Lemma 3.6 implies that $\phi_\delta^n(B) \subset B$, $\forall n \geq 0$, while property (1) of Lemma 3.8 implies that $\phi_\delta^n(A) \subset \phi_\delta^n(B) \subset B$, $\forall n \geq 0$. Property (1) of Lemma 4.3 now implies that $P_\delta(A) \subset B \subset \mathcal{D}(A)$. Since B is compact, $\delta \leq \mu(A)$ and, since δ is arbitrary, the inequality

$$\nu(A) \leq \mu(A)$$

is established.

Now let $\delta < \mu(A)$. Define $B_\varepsilon \equiv \overline{P_\varepsilon(A)}$, and pick $\varepsilon > \delta$ so that $B_\varepsilon \subset \mathcal{D}(A)$ and B_ε is compact. Corollary 4.4 implies that B_ε is an attractor block associated with A. Lemma 3.10 and property (4) of Lemma 4.3 imply that $\beta(B_\varepsilon) \geq \beta(P_\varepsilon(A)) \geq \varepsilon$ and hence that $\nu(A) \geq \varepsilon > \delta$. Since δ is arbitrary, the inequality

$$\nu(A) \geq \mu(A)$$

is established, and the proof is complete.

Conley was interested in the concept of "continuation" of an isolated invariant set in his study of the topological properties persisting under perturbation [**3**]. The remainder of this section is devoted to a brief discussion of the relation between the intensity of an attractor and its continuation properties.

Given two different maps on the same space and an attractor for each map, one attractor is said to "continue immediately" to the other if a common attractor block can be found which is associated with each of the attractors.

Definition: Let A be an attractor for $\phi : X \to X$, and let A' be an attractor for $\psi : X \to X$. The attractor A' is said to be an *immediate continuation* of the attractor A if there exists a subset B of X such that B is both an attractor block for ϕ with associated attractor A and an attractor block for ψ with associated attractor A'.

The notion of "continuation" is obtained from the notion of "immediate continuation" by completing it to a transitive relation. In other words, an attractor for a map is said to "continue" to an attractor for another map if a sequence of maps and attractors can be found, each continuing immediately to the next. Although this notion is very interesting, it will not be pursued further in this paper.

However, the notion of "immediate continuation" of an attractor is closely related to its intensity of attraction. It will be convenient to introduce some notation to be used in the discussion of this relationship. If A is an attractor for the map ϕ, then the intensity depends not only on the set A but also on the map ϕ. If there is any doubt about which map is used in the computation of the intensity, then it will be expicitly indicated. In particular,

$$\beta(S;\phi) \equiv \sup\{\varepsilon : \phi_\varepsilon(S) \subset S\},$$

while

$$\nu(A;\phi) \equiv \sup\{\beta(B;\phi) : B \text{ is an attractor block for } \phi \text{ associated with } A\}.$$

The standard C^0 metric is used on the space of maps,

$$d(\phi,\psi) \equiv \sup_{x\in X} d(\phi(x),\psi(x)),$$

where ϕ and ψ are both maps on X. The following property is an immediate consequence of the definitions.

(5-1) $$|\beta(S;\phi) - \beta(S;\psi)| \leq d(\phi,\psi).$$

Theorem 5.2 *If A is an attractor for the map ϕ and if the map ψ satisfies $d(\phi,\psi) < \nu(A;\phi)$, then there exists an attractor A' for ψ such that A' is an immediate continuation of A.*

Proof: Choose an attractor block B associated with A and satisfying $\beta(B;\phi) > d(\phi,\psi)$. Inequality (5-1) implies that $\beta(B;\psi) > 0$. Theorem 3.11 therefore implies that B is an attractor block for ψ. The associated attractor A' is an immediate continuation of A, which completes the proof.

It is natural to ask whether there is some kind of converse to Theorem 5.2. In other words, given an attractor A for the map ϕ and given an $r > \nu(A;\phi)$, does there exist a map ψ satisfying $d(\phi,\psi) < r$ such that ψ has no attractor which is an immediate continuation of A? The following example shows that the answer, in this generality, is "no". Although it would be interesting to explore the conditions under which the answer is "yes", no such exploration will be undertaken here.

Example 5.3 *Let $B = [-1, 0]$, let $X = B \cup \{1\}$, and let ϕ be the identity map on X. Note that $\phi(B) = B = B^o \subset B^o$ and thus that B is an attractor block for ϕ with associated attractor $A = B$. Note also that $\beta(B; \phi) = d(\phi(B), B^c) = 1$. Therefore, $\nu(A; \phi) = 1$, since B is the only attractor block associated with A. Now let $\psi : X \to X$ satisfy $d(\phi, \psi) < 2$. Since B is connected and since ψ is continuous, either*

(1) $\psi(B) \subset B$, *or*

(2) $\psi(B) \subset \{1\}$.

Note that (2) implies that $\psi(-1) = 1$, which implies that $d(\phi, \psi) \geq 2$, which is a contradiction. Therefore, $\psi(B) \subset B \subset B^o$, which means that B is also an attractor block for ψ. In summary, although $\nu(A; \phi) = 1$, every map ϕ satisfying $d(\phi, \psi) < 2$ has an attractor which is an immediate continuation of A.

6 Discrete Approximations

When a dynamical system is simulated on a computer, a certain kind of approximation is made. A computer has only a finite set of numbers which it can represent. Given a point whose coordinates can be represented, the computer performs some arithmetic and arrives at an approximate image point. The true image point may not be representable, but it is usually safe to assume that the computer's approximation is close to the true one. If everything is working correctly, the computer will always compute the same approximate image point to a given initial point. The ideas discussed in this section are slight modifications of the original ideas found in a paper by Lax [**10**]. The reader is also referred to Rannou [**12**] and Hall [**6**] for further development of the area-preserving case.

To be more precise, and more abstract, the notion of a "net" for the metric space X will be used.

Definition: If Y is a discrete subset of X, if $\delta > 0$, and if $N_\delta(Y) = X$, then Y is called a *δ-net* for X.

Note that Y is a metric space with the same metric d.

Suppose that the set of points representable by the computer is the δ-net Y. Since the computer can represent only those points in Y, an attempt to compute the map $\phi : X \to X$ results in a map $f : Y \to Y$.

Definition: If Y is a δ-net for X, if $\phi : X \to X$, and if $f : Y \to Y$ satisfies

$$d(f(y), \phi(y)) < \varepsilon, \forall y \in Y,$$

then f is called an *ε-approximation* for ϕ.

Note that, if $\delta \leq \varepsilon$, then an ε-approximation to ϕ always exists. Henceforth, this inequality will be assumed. Indeed, since a δ-net is automatically an ε-net for any $\varepsilon \geq \delta$, it will be assumed that an ε-approximation occurs on an ε-net. Thus the phrase "$f : Y \to Y$ is an ε-approximation for $\phi : X \to X$" means that Y is an ε-net for X and the previous definition holds.

In the context of computer simulations, the ε-approximation f is determined from the map $\phi$by the computer arithmetic, the compiler, and the algorithm. The computer will always make the same error if it does the same computation. Thus the simulation of iteration of the original map ϕ on the computer is exactly the iteration of the map f.

If the map ϕ has an attractor, one can ask whether one can expect to see the attractor in a computer simulation. This question can be interpreted as asking whether the map f has an attractor corresponding to the attractor for ϕ. Consider first the analogous question for attractor blocks.

Lemma 6.1 *Let $f : Y \to Y$ be an ε-approximation for $\phi : X \to X$. The following statements are true for every subset S of X.*

(1) $S \cap Y \neq \emptyset$ *and* $\phi_\alpha(S) \subset S \Rightarrow f_{\alpha-\varepsilon}(S \cap Y) \subset S \cap Y$.

(2) $S \cap Y \neq \emptyset \Rightarrow \beta(S \cap Y; f) \geq \beta(S; \phi) - \varepsilon$.

(3) $S \neq \emptyset$ *and* $\beta(S; \phi) > \varepsilon \Rightarrow S \cap Y \neq \emptyset$.

Therefore, if B is an attractor block for ϕ and if $\beta(B; \phi) > \varepsilon$, then $B \cap Y$ is an attractor block for f.

Proof: First, note that the neighborhood $N_\varepsilon(Z)$ is a subset of the ambient space Z. Thus, $f_\gamma(S \cap Y) \subset Y$, for any γ. Recall the convention that $\phi_\gamma(S) \equiv \emptyset$, for $\gamma \leq 0$.

Consider statement (1). If $\alpha \leq \varepsilon$, then $f_{\alpha-\varepsilon}(S \cap Y) = \emptyset \subset S \cap Y$. Assume that $\alpha > \varepsilon$, and let $y \in f_{\alpha-\varepsilon}(S \cap Y)$. By definition, there exists a point $z \in S \cap Y$ such that $d(y, f(z)) < \alpha - \varepsilon$. Then $d(y, \phi(z)) \leq d(y, f(z)) + d(f(z), \phi(z)) < \alpha$, which, since $z \in S$, implies that $y \in N_\alpha(\phi(S)) = \phi_\alpha(S) \subset S$. Therefore, $y \in S \cap Y$, and the proof of statement (1) is complete.

Statement (2) is a consequence of the following inclusion, which is implied by statement (1).

$$\{\alpha : \phi_\alpha(S) \subset S\} \subset \{\alpha : f_{\alpha-\varepsilon}(S \cap Y) \subset S \cap Y\}$$
$$= \{\gamma : f_\gamma(S \cap Y) \subset S \cap Y\} + \varepsilon.$$

Now consider statement (3). The definition of $\beta(S; \phi)$ implies that $N_\varepsilon(\phi(S)) = \phi_\varepsilon(S) \subset S$. Let $x \in \phi(S)$. Since Y is an ε-net, there exists a point $y \in Y$ such that $d(x, y) < \varepsilon$. Therefore, $y \in N_\varepsilon(x) \subset N_\varepsilon(\phi(S)) \subset S$, which means that $Y \cap S \neq \emptyset$, and the proof of statement (3) is complete.

Finally, if B is an attractor block for ϕ, and if $\beta(B; \phi) > \varepsilon$, then, by statement (3), $B \cap Y \neq \emptyset$, and, by statement (2), $\beta(B \cap Y; f) \geq \beta(B; \phi) - \varepsilon > 0$. Therefore, since $B \cap Y$ is compact, Theorem 3.11 implies that $B \cap Y$ is an attractor block for f, and the proof is complete.

It is now clear that, if A is the attractor for ϕ corresponding to the attractor block B, then there is an attractor A' for f corresponding to the attractor block $B \cap Y$. The attractor A' is, in some sense, the computer's best representation of the attractor A.

Definition: Let $f : Y \to Y$ be an ε-approximation for $\phi : X \to X$, let A be an attractor for ϕ, and let A' be an attractor for f. If there exists a subset B of X such that B is an attractor block for ϕ with associated attractor A and such that $B \cap Y$ is an attractor block for f with associated attractor A', then A' is called a *discrete representation* of A.

The following theorem summarizes the previous discussion. It states that, if the intensity of the attractor A exceeds the computer's approximation error, then there exists a discrete representation for A which the computer should be able to find by iterating f.

Theorem 6.2 *Let $f : Y \to Y$ be an ε-approximation for $\phi : X \to X$, and let A be an attractor for ϕ. If $\nu(A; \phi) > \varepsilon$, then there exists an attractor A' for f such that A' is a discrete representation of A.*

Proof: Since $\nu(A; \phi) > \varepsilon$, there exists an attractor block B associated with A and satisfying $\beta(B; \phi) > \varepsilon$. Lemma 6.1 implies that $B \cap Y$ is an attractor block for f. If A' is the attractor associated with $B \cap Y$, then, by definition, A' is a discrete representation of A, and the proof is complete.

Theorem 6.2 gives a sufficient condition for the existence of a discrete representation, but is it a necessary one? In other words, if the intensity is less than the computer error, will the computer be unable to find the attractor? The answer is given by the following theorem, which states that one can find a discrete approximation and an orbit for the discrete approximation which starts in the attractor and leaves the domain of attraction. Of course, the discrete approximation might not be the one that the computer uses, but the theorem shows that, in general, one cannot expect to find attractors with small intensities.

Theorem 6.3 *If A is an attractor for ϕ, if $\varepsilon > \nu(A;\phi)$, and if K is any compact subset of $\mathcal{D}(A)$, then there exists an ε-net Y, an ε-approximation $f : Y \to Y$, and an orbit $(y_0, y_1, \ldots, y_n)$ of f with $y_0 \in A$ and $y_n \notin K$.*

The proof will be given below after the proofs of the following lemmas. Lemma 6.4 makes the observation that an orbit for f is an ε-pseudo-orbit for ϕ. Lemma 6.6 is a converse; it gives conditions under which an ε-pseudo-orbit for ϕ can become an orbit for a suitable ε-approximation.

Lemma 6.4 *If $f : Y \to Y$ is an ε-approximation for $\phi : X \to X$, then an orbit for f is an ε-pseudo-orbit for ϕ.*

Proof: Let $\mathbf{y} = (y_0, y_1, \ldots, y_n)$ be an orbit for f. Then $y_k = f(y_{k-1})$, for $k = 1, 2, \ldots, n$. Since f is an ε-approximation for ϕ,

$$d(y_k, \phi(y_{k-1})) = d(f(y_{k-1}), \phi(y_{k-1})) < \varepsilon, \text{ for } k = 1, 2, \ldots, n,$$

which means that $\mathbf{y}$ is an ε-pseudo-orbit for ϕ. The proof is complete.

An inescapable property of an orbit $(x_0, x_1, \ldots, x_n)$ is that $x_i = x_j$ implies that $x_{i+1} = x_{j+1}$. It will be useful in what is to follow to insist on this "consistency" property for pseudo-orbits.

Definition: An ε-pseudo-orbit $(z_0, z_1, \ldots, z_n)$ for ϕ is called *consistent* if

$$z_i = z_j \Rightarrow z_{i+1} = z_{j+1}, \text{ for } 0 \le i, j \le n-1.$$

Note that the conclusion of Lemma 6.4 can be stated that an orbit for f is a consistent ε-pseudo-orbit for ϕ. Indeed, one may as well always use consistent pseudo-orbits, since inconsistent pseudo-orbits do not go places inaccessible to consistent ones. this sentiment is expressed precisely in Lemma 6.5, but first it is convenient to introduce some notation.

Recall the notation introduced in Section 4: $\Psi_\varepsilon^n(S)$ denotes the set of ε-pseudo-orbits of length n starting on the set S, while $P_\varepsilon^n(S)$ denotes the set of all points on all such pseudo-orbits. Similarly, the notation $\Gamma_\varepsilon^n(S)$ will denote the set of *consistent* ε-pseudo-orbits of length n starting on S, while $Q_\varepsilon^n(S)$ will denote the set of all points on all such pseudo-orbits. More precisely,

$$\Gamma_\varepsilon^n(S) \equiv \{(z_0, z_1, \ldots, z_n) \in \Psi_\varepsilon^n(S) : z_i = z_j \Rightarrow z_{i+1} = z_{j+1}\},$$

$$Q_\varepsilon^n(S) \equiv \{\{z_0, z_1, \ldots, z_n\} : (z_0, z_1, \ldots, z_n) \in \Gamma_\varepsilon^n(S)\}.$$

Note that $n < m \Rightarrow Q_\varepsilon^n(S) \subset Q_\varepsilon^m(S)$.

Lemma 6.5 $Q_\varepsilon^n(S) = P_\varepsilon^n(S)$

Proof: The inclusion $Q_\varepsilon^n(S) \subset P_\varepsilon^n(S)$ is part of the definition. The proof of the opposite inclusion proceeds by induction on n.

The case $n = 0$ is trivial. Assume that

(6-1) $$P_\varepsilon^n(S) \subset Q_\varepsilon^n(S)$$

has been established for $0 \leq n \leq m-1$, and let $x \in P_\varepsilon^m(S)$. Then $x = z_k$, for some $\mathbf{z} = (z_0, z_1, \ldots, z_m) \in \Psi_\varepsilon^n(S)$. If $k < m$, then, by inductive hypothesis, $x \in P_\varepsilon^k(S) \subset Q_\varepsilon^k(S) \subset Q_\varepsilon^m(S)$, and inclusion (6-1) is established for $n = m$. Therefore, assume that $k = m$, that is, assume that $x = z_m$. If $z_i \neq z_j$, for all $i < j$, then $\mathbf{z}$ is consistent, hence $x \in Q_\varepsilon^m(S)$, hence the inclusion is again established. Therefore, assume that $z_i = z_j$, for some $i < j$. Define $\mathbf{w} = (w_0, w_1, \ldots, w_{m-j+1})$ by

$$w_k = \begin{cases} z_k & \text{for} \quad 0 \leq k \leq i, \\ z_{k-i+j} & \text{for} \quad i < k \leq m-j+i. \end{cases}$$

Note that $\mathbf{w}$ is an ε-pseudo-orbit and hence that $x = z_k = w_{m-j+i} \in P_\varepsilon^{m-j+i}(S)$. Therefore, by inductive hypothesis, $x \in Q_\varepsilon^{m-j+i}(S) \subset Q_\varepsilon^m(S)$, and inclusion (6-1) is established for $n = m$. The proof is complete.

Lemma 6.6 *Let* $\mathbf{z}$ *be a consistent* ε*-pseudo-orbit for* $\phi : X \to X$*. There exists an* ε*-net* Y *for* X *and an* ε*-approximation* $f : Y \to Y$ *for* ϕ *such that* $\mathbf{z}$ *is an orbit for* f*.*

Proof: Let $\mathbf{z} = (z_0, z_1, \ldots, z_n)$, define $Y_0 \equiv \{z_0, z_1, \ldots, z_{n-1}\}$, and extend $Y_0 \cup \{z_n\}$ to an ε-net Y for X. Define $f : Y \to Y$ as follows. If $y \notin Y_0$, then choose $f(y)$ satisfying $d(f(y), \phi(y)) < \varepsilon$. The fact that Y is an ε-net for X insures that this choice can be made. If $y = z_k \in Y_0$, then take $f(y) = z_{k+1}$. The consistency of $\mathbf{z}$ insures that f is well-defined. Since $\mathbf{z}$ is an ε-pseudo-orbit for ϕ, $d(f(z_k), \phi(z_k)) = d(z_{k+1}, \phi(z_k)) < \varepsilon$, for $k = 0, 1, \ldots, n-1$. Therefore, f is an ε-approximation for ϕ. By construction, $\mathbf{z}$ is an orbit for f, and the proof is complete.

Proof of Theorem 6.3: By Theorem 5.1, the intensity of A is equivalent to its chain intensity. Therefore, either $\overline{P_\varepsilon(A)}$ is not a subset of $\mathcal{D}(A)$ or it is not compact. In either case, there exists a point $z \in \overline{P_\varepsilon(A)} \cap K^c$, which implies the existence of a point $w \in P_\varepsilon(A) \cap K^c$. By definition of $P_\varepsilon(A)$, $w \in P_\varepsilon^m(A)$, for some m. Lemma 6.5 implies that $w \in Q_\varepsilon^m(A)$. Therefore, there exists a consistent ε-pseudo-orbit $\mathbf{z} = (z_0, z_1, \ldots, z_n)$ such that $z_0 \in A$ and $z_n = w$. Lemma 6.6 implies the existence of an ε-net Y for X and an ε-approximation $f : Y \to Y$ for ϕ such that $\mathbf{z}$ is an orbit for f. Since $z_0 \in A$ and $z_n = w \notin K$, the proof is complete.

7 Subsystems

When the map ϕ is restricted to a positively invariant subset W of X, the restriction is a dynamical system in its own right, often called a subsystem of ϕ. It will be shown in this section that attractors restrict to subsystems in a nice way.

Recall the notation introduced in Section 5 for the intensity when the dependence on the map is explicitly denoted. The analogous notation will also be used for the omega limit set, namely,

$$\tau_n(S;\phi) \equiv \bigcup_{k \geq n} \phi^k(S),$$

$$\omega(S;\phi) \equiv \bigcap_{n \geq 0} \overline{\tau_n(S;\phi)}.$$

It should be noted that the closure operation refers to closure in the ambient space X. The restriction of ϕ to W will be denoted $\phi|_W$.

Lemma 7.1 *If W is closed and positively invariant under ϕ, if $S \subset W$, and if $\psi \equiv \phi|_W$, then $\omega(S;\psi) = \omega(S;\phi)$.*

Proof: Since $S \subset W$, $\psi(S)$ is defined and equals $\phi(S)$. Therefore, $\psi^k(S) = \phi^k(S)$, for all k, which implies that $\tau_n(S;\psi) = \tau_n(S;\phi)$, for all n. Since W is closed, the closure in W of a subset of W is identical to its closure in X. Therefore, $\overline{\tau_n(S;\psi)} = \overline{\tau_n(S;\phi)}$, which implies that $\omega(S;\psi) = \omega(S;\phi)$, and the proof is complete.

Lemma 7.2 *If W is positively invariant under ϕ, if $S \subset X$, and if $\psi \equiv \phi|_W$, then the following properties hold.*
(1) $\psi_\varepsilon(S \cap W) \subset \phi_\varepsilon(S) \cap W$.
(2) $\beta(S \cap W; \psi) \geq \beta(S;\phi)$.

Proof: It is essential to distinguish between a neighborhood in X and a neighborhood in W. To this end, write

$$N_\varepsilon(S;X) \equiv \cup_{x\in S}\{y \in X : d(x,y) < \varepsilon\},$$

and note that

$$N_\varepsilon(S \cap W; X) \subset N_\varepsilon(S;X) \cap W.$$

Note also that, since W is positively invariant,

$$\psi(S \cap W) = \phi(S \cap W) \subset \phi(S) \cap \phi(W) \subset \phi(S) \cap W.$$

Therefore,

$$\psi_\varepsilon(S \cap W) \equiv N_\varepsilon(\psi(S \cap W); W) \subset N_\varepsilon(\phi(S) \cap W; W)$$

$$\subset N_\varepsilon(\phi(S); X) \cap W = \phi_\varepsilon(S) \cap W,$$

which establishes property (1)

Now write $S' \equiv S \cap W$, and suppose that $\phi_\varepsilon(S) \subset S$, for some ε. Property (1) then implies that

$$\psi_\varepsilon(S') \subset \phi_\varepsilon(S) \cap W \subset S \cap W = S'.$$

Therefore,

$$\{\varepsilon : \phi_\varepsilon(S) \subset S\} \subset \{\varepsilon : \psi_\varepsilon(S') \subset S'\}.$$

Taking the supremum of both sides establishes property (2) and completes the proof.

Lemma 7.3 *If W is closed and positively invariant under ϕ, if B is an attractor block for ϕ, if $B' \equiv B \cap W \neq \emptyset$, and if $\psi \equiv \phi|_W$, then B' is an attractor block for ψ and*

$$\beta(B';\psi) \geq \beta(B;\phi).$$

Proof: Property (2) of Lemma 7.2 implies the inequality. By hypothesis, B' is nonempty. Since W is closed and B is compact, B' is compact. Theorem 3.11 implies that $\beta(B;\phi) > 0$, hence that $\beta(B';\psi) > 0$, and hence that B' is an attractor block. The proof is complete.

Theorem 7.4 *If W is closed and positively invariant under ϕ, if A is an attractor for ϕ, if $A' \equiv A \cap W \neq \emptyset$, and if $\psi \equiv \phi|_W$, then A' is an attractor for ψ. Furthermore,*
(1) $\mathcal{D}(A') = \mathcal{D}(A) \cap W$, *and*
(2) $\nu(A';\psi) \geq \nu(A;\phi)$.

Proof: Since A is an attractor for ϕ, there exists a neighborhood U of A in X satisfying $\omega(U;\phi) = A$. If $V \equiv U \cap W$, then V is a neighborhood of A' in W. Properties (1) and (2) of Lemma 1.9 imply that

$$A' = \omega(A';\phi) \subset \omega(V;\phi) \subset \omega(U;\phi) = A.$$

Lemma 7.1 therefore implies that $A' \subset \omega(V;\psi) \subset A$. Since $\omega(V;\psi) \subset W$, it follows that $\omega(V;\psi) \subset A \cap W = A'$, which establishes that A' is an attractor for ψ.

The definition of domain of attraction is given in Section 2 and can be restated in the present context as follows.

$$\mathcal{D}(A) = \{x \in X : \emptyset \neq \omega(x,\phi) \subset A\}.$$

$$\mathcal{D}(A') = \{x \in W : \emptyset \neq \omega(x,\psi) \subset A'\}.$$

In view of Lemma 7.1, the first formula implies that

$$\mathcal{D}(A) \cap W = \{x \in W : \emptyset \neq \omega(x,\psi) \subset A\}.$$

Since $\omega(x,\psi) \subset W$ and since $A' = A \cap W$, equality (1) follows.

Now let

$$\Gamma = \{\beta(B;\phi) : B \text{ is an attractor block associated with } A\},$$

$$\Gamma' = \{\beta(B';\psi) : B' \text{ is an attractor block associated with } A'\}.$$

Since $\nu(A;\phi) = \sup\Gamma$ and $\nu(A';\psi) = \sup\Gamma'$, inequality (2) is established once it is shown that

(7-1) $$\forall\gamma \in \Gamma, \exists\gamma' \in \Gamma' \text{ such that } \gamma' \geq \gamma.$$

To this end, let B be an attractor block associated with A, and let $\gamma \equiv \beta(B;\phi)$. Since $\emptyset \neq A \cap W \subset B \cap W = B'$, Lemma 7.3 implies that B' is an attractor block and that $\beta(B';\psi) \geq \beta(B;\phi)$. Therefore, $\gamma' = \beta(B';\psi) \geq \gamma$, which establishes statement (7-1) and completes the proof.

Theorem 7.5 *If A is an attractor for ϕ and if A' is an attractor for $\phi|_A$, then A' is an attractor for ϕ.*

Proof:All neighborhoods in this proof will be neighborhoods in the space X unless otherwise explicitly indicated. Similarly, all omega limit sets will be for the map ϕ unless otherwise indicated.

Write $\psi \equiv \phi|_A$. Theorem 2.1 (2) implies the existence of a closed neighborhood K' of A' in A satisfying

(7-2) $$\omega(K';\psi) = A'.$$

Extend K' to a closed neighborhood K of A' in X, and note that $K' = K \cap A$. Theorem 3.2 implies the existence of an attractor block B' for ψ associated with A' and such that B' is in the interior of K' relative to A. Note that $B' \subset K^o$, where now the interior is relative to X. Choose ε satisfying $0 < \varepsilon < \beta(B';\phi)$, and choose $\delta > 0$ satisfying

(7-3) $$N_\delta(B') \subset K,$$

(7-4) $$\phi(N_\delta(B')) \subset N_\varepsilon(\phi(B')),$$

(7-5) $$\varepsilon + \delta < \beta(B';\psi).$$

The two inclusions follow from Lemmas 3.3 and 1.4. Theorem 2.1 (3) now implies the existence of a compact neighborhood G of A satisfying

$$G \subset N_\delta(A), \text{ and } \omega(G) = A.$$

Now let

$$S \equiv \overline{N_\delta(B')} \cap G,$$

and note that S is a compact neighborhood of A'. Note also that

$$\omega(S) \subset \omega(G) = A.$$

It will be shown below that S is positively invariant. Lemma 1.9 (4) and inclusion (7-3) then imply that

$$\omega(S) \subset S \subset \overline{N_\delta(B')} \subset K,$$

and hence that

$$\omega(S) \subset K \cap A = K'.$$

Since S is a compact positively invarant set, Lemma 1.10 implies that $\omega(S)$ is a compact invariant set, which, by Lemma 1.9 (2), implies that $\omega(\omega(S)) = \omega(S)$. Lemma 7.1 and equation (7-2) imply that $\omega(K') = A'$, from which it follows that

$$\omega(S) \subset A'.$$

Since A' is compact and invariant and since $A' \subset S$, it follows that $A' \subset \omega(S)$ and hence that $A' = \omega(S)$. Therefore, by definition, A' is an attractor.

It remains to show only that S is positively invariant. Since G is positively invariant, it follows that

(7-6) $$\phi(S) \subset \phi(G) \subset G \subset N_\delta(A).$$

Inclusion (7-4) implies that

$$\phi(S) \subset \phi(N_\delta(B')) \subset N_\varepsilon(\phi(B')),$$

which, with the previous inclusion implies that

(7-7) $$\phi(S) \subset N_\varepsilon(\phi(B')) \cap N_\delta(A).$$

It will be shown below that

(7-8) $$N_\varepsilon(\phi(B')) \cap N_\delta(A) \subset N_\delta(B'),$$

which, with inclusions (7-6) and (7-7), implies that

$$\phi(S) \subset N_\delta(B') \cap G \subset S$$

and hence that S is positively invariant.

The proof will be complete once inclusion (7-8) is established. To this and, let

$$y \in N_\varepsilon(\phi(B')) \cap N_\delta(A).$$

There exist a $z \in \phi(B')$ and an $x \in A$ such that

$$d(y, z) < \varepsilon \text{ and } d(y, x) < \delta.$$

Therefore, $d(x,z) < \varepsilon + \delta$, which implies that $x \in N_{\varepsilon+\delta}(\phi(B'))$. Since $x \in A$, it follows that $x \in \psi_{\varepsilon+\delta}(B')$. But inequality (7-5) implies that $\psi_{\varepsilon+\delta}(B') \subset B'$, which implies that $x \in B'$. Since $d(y,x) < \delta$, it follows that

$$y \in N_\delta(B'),$$

which establishes inclusion (7-8) and completes the proof of Theorem 7.5.

8 Examples

Sometimes it is possible to compute the intensity exactly, as the following example shows.

Example 8.1 *Let $X = [0,b]$, and let $\psi : X \to X$ satisfy*
(1) $\psi(0) = 0$ *and* $\psi(b) = b$,
(2) $\psi(x) < x$, *for* $0 < x < b$, *and*
(3) ψ *is strictly increasing on* X.
The set $A \equiv \{0\}$ is an attractor satisfying

$$\nu(A) = \max\{x - \psi(x) : 0 \le x \le b\}.$$

Proof: It is a standard exercise to show that A is an attractor with $\mathcal{D}(A) = [0,b)$. Let

$$M \equiv \max\{x - \psi(x) : 0 \le x \le b\}.$$

Note that $B_t \equiv [0,t]$ is an attractor block associated with A for $0 < t < b$ and that $\beta(B_t) = t - \psi(t)$. Therefore, $\nu(A) \ge \sup\{\beta(B_t) : 0 < t < b\} = M$, which establishes the inequality

$$\nu(A) \ge M.$$

Now consider the sequence $(z_0, z_1, \ldots, z_n)$, where $z_0 = 0$, and $z_k = \delta + \psi(z_{k-1})$, for $k = 1, 2, \ldots, n$. If $\delta > M$, then $z_j \ge z_{j-1} + (\delta - M)$, which means that $z_j \ge j(\delta - M)$. Let k satisfy $z_j < b$, for $0 \le j < k$, and $z_k \ge b$. Define $\mathbf{w} = (w_0, w_1, \ldots, w_n)$ by

$$w_j = \begin{cases} z_j & \text{for} \quad 0 \le j < k, \\ b & \text{for} \quad j = k. \end{cases}$$

Note that **w** is an ε-pseudo-orbit for any $\varepsilon > \delta$ and that **w** leaves any compact subset of $\mathcal{D}(A)$. Since δ can be chosen arbitrarily close to M, this statement implies that

$$\nu(A) = \mu(A) \leq M,$$

and the proof is complete.

It is often of sufficient interest to find only an upper bound for the intensity of an attractor. A good upper bound can be demonstrated for the following example.

Example 8.2 *Let $X = \mathbf{R}$, and let $\phi : \mathbf{R} \to \mathbf{R}$ satisfy*

(1) $\phi(0) = 0$,

(2) *there exists $b' < 0$ such that $\phi(x) > x$ for $x \in (b', 0)$,*

(3) *there exists $b > 0$ such that $\phi(b) = b$ and such that $\phi(x) < x$, for $x \in (0, b)$, and*

(4) *ϕ is strictly increasing on $[0, b]$.*

The set $A \equiv \{0\}$ is an attractor satisfying

$$\nu(A) \leq \max\{x - \phi(x) : 0 \leq x \leq b\}.$$

Proof: A standard argument shows that A is an attractor and that $W \equiv [0, b]$ is an invariant set. The estimate can be reduced to the previous example by letting $\psi \equiv \phi|_{[0,B]}$. Theorem 7.4 then implies that $\nu(A; \phi) \leq \nu(A; \psi)$. But the previous example shows that

$$\nu(A; \psi) = \max\{x - \psi(x) : 0 \leq x \leq b\}$$

$$= \max\{x - \phi(x) : 0 \leq x \leq b\},$$

which completes the proof.

The next example illustrates another case where an upper bound on the intensity can be given. It will be used also in the computation in the examples below.

Example 8.3 *Let X be an interval containing 0 in its interior, and let $\phi : X \to X$ satisfy*

(1) $\phi(0) = \phi'(0) = 0$,

(2) *$\phi''(0) \geq c > 0$, and*

(3) *$\phi'''(x) \geq -\frac{3}{8}c^2$, for all $x \in X$.*

Then $A = \{0\}$is an attractor with $\nu(A) \leq 16/27c$.

Proof: First note that, for all $x \in X$,

(8-1) $$\phi'(x) \geq cx - \tfrac{3c^2}{16}x^2,$$

(8-2) $$\phi(x) \geq \tfrac{c}{2}x^2 - \tfrac{c^2}{16}x^3.$$

Formula (8-2) implies that $\phi(4/c) \geq 4/c$. Therefore, there exists a positive number $b \leq 4/c$ such that

$$\phi(b) = b, \text{ and}$$

$$\phi(x) < x, \text{ for } 0 < x < b.$$

Formula (8-1) implies that

$$\phi'(x) > 0, \text{ for } 0 < x < b.$$

It follows that ϕ meets the conditions of Example 8.2 and hence that $A = \{0\}$ is an attractor satisfying

$$\nu(A) \leq \max\{x - \phi(x) : 0 \leq x \leq b\}.$$

Now define

$$g(x) \equiv x - \frac{c}{2}x^2 + \frac{c^2}{16}x^3,$$

and note that inequality (8-2) implies that

$$x - \phi(x) \leq g(x).$$

Therefore,

$$\nu(A) \leq \max\{g(x) : 0 \leq x \leq b\} \leq \max\{g(x) : 0 \leq x \leq 4/c\}.$$

An elementary calculus exercise shows that this last quantity is equal to $16/27c$, and the proof is complete.

Example 8.4 *Let $X = \mathbf{R}$, and let $\phi_c : \mathbf{R} \to \mathbf{R}$ be the general quadratic map in standard form:*

$$\phi_c(x) = x^2 + c.$$

There is a unique value of c, which happens to be close to -2, for which there is a superattracting orbit of period q with itinerary $CLRR \cdots R$. In other words, there is a periodic orbit $A \equiv \{x_0, x_1, \ldots, x_{q-1}\}$, with $x_0 = 0$, $x_1 < 0$ and $x_k > 0$, for $k = 2, 3, \ldots, q-1$. Using the techniques of the previous examples, one can show that

$$\nu(A) \sim 16^{-q}, \text{ for large } q.$$

Note that the intensity is of the order 2^{-64} when q is 16. Therefore, this particular family of attractors will be extremely difficult to detect by direct computer simulation for even relatively modest periods.

Example 8.5 *Let $X = \mathbf{R}^2$, and let ϕ_a be the time 1 map of the vector field*

$$\dot{r} = ar - r^3,$$

$$\dot{\theta} = r^{q-2} \sin q\theta,$$

where (r, θ) are polar coordinates on $\mathbf{R}^2$. For positive values of a, this map has two attractors,

$$A \equiv \{(r, \theta) : r = \rho\} \text{ and}$$

$$A' \equiv \{(r, \theta) : r = \rho \text{ and } \theta = (2j-1)\pi/q, j = 1, 2, \ldots, q\},$$

where $\rho = \sqrt{a}$. Note that A is an invariant circle, A' is a set of q fixed points, and $A' \subset A$. Using the techniques of the earlier examples, one can estimate that

$$\nu(A) \sim \rho^3 \text{ and}$$

$$\nu(A') \sim \rho^{q-1},$$

for small a.

This example appears to be artificial, but it is related to supercritical Hopf bifurcation for maps of the plane. The attractor A corresponds to the invariant circle, while the attractor A' corresponds to the periodic sink with rotation number p/q. One can see that, while the invariant circle is not too difficult to detect with direct computer simulations, even modestly high resonances pose a problem. For example, with 64 bit arithmetic, one can reasonably expect to detect an invariant circle with a radius of 2^{-20}. However, one would expect to have difficulty detecting a periodic sink of period 33 for a radius less than $1/4$. Experience has shown that these resonances are indeed difficult to find with direct computer simulations [**1**].

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